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Outperforming a Stochastic Benchmark Under Borrowing and Rectangular Constraints

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Abstract

In this paper, we extend Browne (1999) into the cases of borrowing and rectangular constraints. To address the constraints, we construct an auxiliary market by employing the framework of Cvitanic and Karatzas (1992). Under this framework, we show how problems concerned with survival, growth and goal reaching with respect to a benchmark can be solved by applying the techniques of stochastic optimal control. From the solutions, we see that, while the results under the borrowing constraints may be obtained analytically, the results under the rectangular constraints require a computational procedure. Furthermore, we provide an analysis by using our solutions to see the effect of constraints and changes in key parameters on the optimal results. We also conduct a numerical analysis with three examples to clarify further the effect of the constraints on the investment behaviour of an investor trading in favorable markets.

JEL Classification: C61, D81, G11, G22.
Keywords: Hamilton-Jacobi-Bellman Equations; Stochastic optimal control; Portfolio selection; Borrowing constraints; Rectangular constraints; Benchmark; Auxiliary Market.

1 Introduction

In this paper, we examine the performance of a portfolio manager with respect to an exogenously given stochastic benchmark under the borrowing and rectangular constraints. To this end, we extend Browne (1999) and show how to find the optimal investment strategies under the constraints we consider. To address the constraints, we use the framework of Cvitanic and Karatzas (1992) (see also Karatzas and Shreve (1998)), which is a duality method based technique that allows to maximize utility in incomplete markets. Briefly, the technique deals with incompleteness by completing the market via so-called fictitious parameters under an auxiliary market. The fictitious parameters act as Lagrange multipliers and optimal solutions may in turn be found upon specifying them.

We obtain the specifications of these fictitious parameters as well as the optimal portfolio strategies to outperform a benchmark by employing the techniques of stochastic optimal control. In this respect, our approach for solving the problems is different from the convex duality argument of Cvitanic and Karatzas (1992) under the martingale approach. This approach, introduced by Pliska (1986), is an alternative to the dynamic programming principle applied by Merton (1971) (see also Merton (1969)) to a general class of utility functions. In our study, we show how the dynamic programming principle can be applied to solve problems defined

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under the borrowing and rectangular constraints. We provide explicit specifications for the optimal value function and strategy in each problem. However, as for the optimal fictitious parameters, we see that, while the results under the borrowing constraints have analytical forms, the results under the rectangular constraints require a computational step. In all cases, we also provide the implications of constraints and changes in key parameters on the optimal results.

For the types of constraints we consider, some of the relevant studies[3] our work is in line with are Tepla (2000) for considering the case of an investor facing borrowing constraints or no-shorting constraints or both, Detemple et al. (2005) for examining the portfolio choice of an investor with partially hedgeable risk under the rectangular constraints, Jin and Zhang (2013) for studying the dynamic portfolio in jump diffusion models under borrowing, trading and short-selling prohibitions and finally Yener (2015) for finding the optimal portfolio choice of an investor maximizing growth, survival and goal reaching under borrowing prohibition. This study mainly differs from Tepla (2000), Detemple et al. (2005) and Jin and Zhang (2013) for the model, problems and solution methodology it considers while it differs from Yener (2015) for the model and constraints. The constraints we employ cover all the constraints considered in the aforementioned studies.

The model we use, as shown in Browne (1999), consists of a traded portfolio expressed in the units of an exogenously given stochastic benchmark. The traded portfolio consists of one risk free asset and N risky assets whose dynamics are given as geometric Brownian motions. The dynamics of the benchmark is similar to those of the investor's portfolio, however, it also involves an uncommon exogenously given risk factor. Due to the existence of an uncommon factor, the market becomes incomplete. Yet via straightforward application of the techniques of stochastic optimal control, Browne provides analytical solutions of all problems which are namely (i) the minimization (maximization) of the expected time to beat (stay above) a benchmark, (ii) the maximization of the probability that the traded portfolio beats the benchmark before incurring a certain level of shortfall, (iii) the maximization (minimization) of expected discounted reward (penalty) from reaching a certain level. Here, the minimization of the expected time and the maximization of reward are classified as goal reaching problems that are concerned with the growth of a traded portfolio. The maximization of the time to stay above a certain level of shortfall, the maximization of the probability of success and the minimization of penalty are classified as survival problems.

To solve the problems, we first construct the traded portfolio and introduce the benchmark process in Section 2.1. We then define the problems along with the benchmarked portfolio process in Section 2.2. To address the constraints, we construct an auxiliary market as in Cvitanic and Karatzas (1992) in Section 2.3. In that section, we introduce the fictitious parameters that allow us to relax the constraints, and solve the problems as if there are no constraints at all. After the introduction of the auxiliary market, we proceed to the problem formulation in Section 3 and show the main theorem (Theorem 3.1) from which the proofs of the problems follow. The proof of Theorem 3.1 is done with the application of the dynamic programming principle (see Fleming and Soner (1993)) under the auxiliary market. After introducing Theorem 3.1, the first problem related to the minimization (maximization) of the expected time the benchmarked portfolio process reaches (stay above) an upper (lower) level is solved in Section 4. We solve the second and third problems in Sections 5 and 6 respectively. We remind that the

[3]See also Fleming and Zariphopoulou (1991), Zariphopoulou (1994) and Vila and Zariphopoulou (1997) for trading a single risky asset and a riskless asset under constraints related to borrowing, Xu and Shreve (1992), He and Pearson (1991), Fleming and Zariphopoulou (1991) for short-selling constraints.

second problem is the probability that the benchmarked portfolio process reaches an upper level without first hitting a lower level and the third problem is the maximization (minimization) of the expected discounted reward (penalty). Finally, we provide three numerical examples in Section 7 as complementary analysis to make better sense of our findings and understand how changes in key parameters affect the results. We conclude with Section 8.

2 The Assets and The Model

2.1 Financial Assets

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbb{P})$ supporting $(N+1)$-dimensional standard Brownian motion $B(t) = (B_1(t), \ldots, B_{N+1}(t))^{'}$ [4], $0 \leq t < \infty$, and $\{\mathcal{F}_t\}_{0\leq t<\infty}$ is the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^B := \sigma\{B(u) \mid u \leq t\}$. Here, N Brownian motions represent the uncertainties in the risky assets and $(N+1)$th Borwnian motion represents the risk of the uncommon factor of an exogenously given stochastic benchmark process.

We assume that an investor continuously trades one risk-free and N risky assets in a Black-Scholes type financial market with no transaction costs. We represent the drivers of the financial assets by

$$\begin{aligned} \mathrm{d}V_0(t) &= rV_0(t)\mathrm{d}t; && (2.1) \\ \mathrm{d}S_i(t) &= S_i(t)\left[\mu_i \mathrm{d}t + \sum_{j=1}^{N} \sigma_{ij}\mathrm{d}B_j(t)\right] \quad \text{for} \quad i = 1, \ldots, N, && (2.2) \end{aligned}$$

where Equation (2.1) is the risk-free money market account process with a constant riskless rate $r \geq 0$. Equation (2.2) is the risky assets' price processes. We call risky assets stocks and assume that μ_i and volatility $\sigma_{ij} > 0$, for $i, j = 1, \ldots, N$, are constants.

When trading in the market, the investor invests some proportion of her wealth in stocks and the remainder in the money market account. The proportions of wealth invested in the risky assets at time t is denoted by a vector of control processes $\boldsymbol{w}(t) := (w_1(t), \ldots, w_N(t))^{'}$. Mainly, $\boldsymbol{w}(\cdot)$ is called an investment strategy and we denote the admissibility of an investment strategy for an initial capital x by $\boldsymbol{w}(\cdot) \in \mathcal{A}(x)$. That is, $\mathcal{A}(x)$ is the set of admissible strategies. We say that $\boldsymbol{w}(t) \in \mathcal{A}(x)$ if $\boldsymbol{w}(t)$ is $\{\mathcal{F}_t\}$-progressively measurable, satisfies $\int_0^t \|\boldsymbol{w}(s)\|^2 ds < \infty$ $\mathbb{P}$-almost surely for $t < \infty$. Then, the self-financing wealth process associated to an admissible strategy is the solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}X^{\boldsymbol{w}}(t) &= X^{\boldsymbol{w}}(t)\left[r\mathrm{d}t + \boldsymbol{w}^{'}(t)(\mu - r\underset{\sim}{1})\mathrm{d}t + \boldsymbol{w}^{'}(t)\sigma\mathrm{d}B^{\bullet}(t)\right]; \\ X(0) &= x, && (2.3) \end{aligned}$$

where $\mu = (\mu_1, \ldots, \mu_N)^{'}$, $\underset{\sim}{1} = (1, \ldots, 1)^{'}$, $\sigma = (\sigma_1, \ldots, \sigma_N)^{'}$ with $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{iN})$ for $i = 1, \ldots, N$ and finally $B^{\bullet}(t) = (B_1(t), \ldots, B_N(t))^{'}, 0 \leq t < \infty$. We observe that the closed form solution of (2.3) always gives $X^{\boldsymbol{w}}(t) > 0$ for $t < \infty$. Therefore, under proportional investment strategies, the portfolio process never hits zero in finite time.

[4] The sign $'$ denotes the transpose if not indicated otherwise. We will see in the sequel that, we also use $'$ and $''$ to denote the first and second derivatives of a function with respect to its argument.

Finally, we consider an exogenously given benchmark process $Y(\cdot)$, which is the solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}Y(t) &= Y(t)[\alpha \mathrm{d}t + b^{'}\mathrm{d}B^{\bullet}(t) + \beta \mathrm{d}B^{N+1}(t)]; \\ Y(0) &= y, \end{aligned} \tag{2.4}$$

where α and β are constants and $b^{'}$ denotes a constant column vector; $b = (b_1, b_2, \ldots, b_N)^{'}$. Furthermore, $B^{N+1}(\cdot)$ is the additional standard Brownian motion associated to the uncontrollable source of risk[5].

The specification in (2.4) is the benchmark process studied by Browne (1999). We observe that the benchmark process is partially correlated with the wealth process $X^{\boldsymbol{w}}(\cdot)$ for $\beta \neq 0$. This inequality allows for further generalization and makes the benchmark process be interpreted in various forms such as inflation or exchange rate, the price process of a non-traded asset, idiosyncratic risk associated with the management of a benchmark portfolio and such. However, when $\beta = 0$, then the benchmark process can be interpreted for example as a benchmark portfolio process (For the case when $\beta = 0$ and the maturity is $T < \infty$ we refer the reader to Browne (1997)). As noted in Browne (1999), with additional Brownian motion the market becomes incomplete as there are more risk factors than the number of liquidly traded financial assets.

2.2 Performance Measures & The Benchmarked Portfolio

The benchmark process is used to measure the relative goal and shortfall performance of an investor. For fixed L, U, with $yL < x < yU$, we posit that the relative goal is achieved when $X^{\boldsymbol{w}}(t) = UY(t)$ for $t > 0$, and the relative shortfall happens when $X^{\boldsymbol{w}}(t) = LY(t)$ for $t > 0$. The problems associated with the performance measurement are then: (i) Minimizing the expected time of beating the benchmark or maximizing the expected time until shortfall; (ii) Maximizing the probability of reaching a predetermined higher wealth level before incurring a shortfall; (iii) Maximizing/Minimizing the expected discounted reward/penalty.

Our goal is to solve the problems under the borrowing and rectangular constraints. To this end, we first call for a new controlled stochastic process $Z^{\boldsymbol{w}}(t) := X^{\boldsymbol{w}}(t)/Y(t)$. We observe that $Z^{\boldsymbol{w}}(\cdot)$ is a ratio process which is equivalent to the traded portfolio expressed in units of the benchmark process. In other words, it is the benchmarked portfolio process. Its value clearly reflects how well an investor performs relative to a benchmark. By the application of the Itô's formula, we then write the dynamics of the benchmarked portfolio process as

$$\begin{aligned} \mathrm{d}Z^{\boldsymbol{w}}(t) &= Z^{\boldsymbol{w}}(t)\left[\left(\hat{r} + \boldsymbol{w}^{'}(t)(\hat{\mu} - r\underset{\sim}{1})\right)\mathrm{d}t + (\boldsymbol{w}^{'}(t)\sigma - b^{'})\mathrm{d}B^{\bullet}(t) - \beta \mathrm{d}B^{(N+1)}(t)\right]; \\ Z(0) &= z, \end{aligned} \tag{2.5}$$

where $\hat{r} = r + \|b\|^2 - \alpha + \beta^2$ and $\hat{\mu} = \mu - \sigma b$.

Next, we construct an auxiliary market endowed with fictitious assets under the context of Cvitanic and Karatzas (1992) (see also Karatzas and Shreve (1998)) to address the constraints. The substance of the auxiliary market is to provide a mathematical ground that allows us to trade freely as if there are no constraints. Within this framework, we then extend the generalized version of the problems after we introduce the auxiliary market in the next section.

[5] This is the uncommon exogenously given risk factor mentioned in the Introduction.

2.3 The Auxiliary Market

An auxiliary market is an augmentation of the constrained market with the use of fictitious parameters, specifically the dual processes so-called Lagrange multipliers. To provide a background, we follow Cvitanic and Karatzas (1992) and Karatzas and Shreve (1998).

First, we define a closed convex set $\mathcal{K} \neq \emptyset$ of $\mathbb{R}^N$. This is the constraint set that contains proportional investment strategies in N risky assets. Next, for a given $\mathcal{K}$, we define the *support function* of the convex set $-\mathcal{K}$ by

$$\delta(\nu \mid \mathcal{K}) \equiv \delta(\nu) := \sup_{\boldsymbol{w} \in \mathcal{K}} (-\boldsymbol{w}' \nu) : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}, \quad \text{for } \nu \in \mathbb{R}^N.$$

The support function, finite on its effective domain, is defined by

$$\tilde{\mathcal{K}} := \{\nu \in \mathbb{R}^N \mid \delta(\nu) < \infty\}.$$

Here, $\tilde{\mathcal{K}}$ is the barrier cone of $-\mathcal{K}$ and we assume that $\tilde{\mathcal{K}}$ contains the origin on $\mathbb{R}^N$. The assumption arises from the fact that $\nu = \mathbf{0}_N$[6] when the constraints are not binding. Then, from the specification of the support function, it follows that $\delta(0) = 0$. In fact, $\delta(\nu) \geq 0$ $\forall \nu \in \mathbb{R}^N$ and $\delta(\nu) + \boldsymbol{w}' \nu \geq 0$, $\forall \nu \in \tilde{\mathcal{K}}$ if and only if $\boldsymbol{w} \in \mathcal{K}$.

Furthermore, we let $\nu := \{\nu(t) \mid 0 \leq t < \infty\}$ be the vector of $\{\mathcal{F}_t\}$-progressively measurable fictitious processes in the space $\mathcal{D}$ of fictitious processes taking values in $\tilde{\mathcal{K}}$. With the use of these processes, we construct an auxiliary market by relaxing the constraints and allowing investment to be done as if there are no constraints at all. Once the optimal investment strategy in the auxiliary market is found, we then find the optimal result under the constrained market by optimizing over $\nu(\cdot) \in \mathcal{D}$. In this way, we find a particular value $\nu^*(t)$ that makes $\boldsymbol{w}(t) \in \mathcal{K}$ for $t < \infty$. More clearly, with $\nu^*(\cdot)$, the unconstrained investment strategy in the auxiliary market is the same as the investment strategy in the constrained market[7]. Therefore, $\nu^*(\cdot)$ is equivalent to the value of the proportional investment amount that violates the constraints.

We consider two types of constraints which are namely the borrowing and rectangular constraints[8]. For these constraints, we specify $\mathcal{K}$, $\tilde{\mathcal{K}}$, and $\delta(\nu)$ in the following way:

Borrowing constraints: $\mathcal{K} = \{\boldsymbol{w} \in \mathbb{R}^N \mid \sum_{i=1}^N w_i \leq \kappa\}$ for some $\kappa \geq 1$. Then, $\tilde{\mathcal{K}} = \{\nu \in \mathbb{R}^N \mid \nu_1 = \ldots = \nu_N \leq 0\}$ and $\delta(\nu) = -\kappa \nu_1$ on $\tilde{\mathcal{K}}$;

Rectangular constraints: $\mathcal{K} = I_1 \times \ldots I_N$ with $I_n = [l_n, u_n]$, $-\infty \leq l_n \leq 0 \leq u_n \leq \infty$ and with the understanding that I_n is open on the right (respectively, left) if $u_n = \infty$ (respectively, $l_n = -\infty$). Then $\tilde{\mathcal{K}} = \mathbb{R}^N$, $\delta(\nu) = -\sum_{n=1}^N (l_n \nu_n^+ - u_n \nu_n^-)$ on $\tilde{\mathcal{K}}$ if all l_n and u_n are finite. In general, $\tilde{\mathcal{K}} = \{\nu \in \mathbb{R}^N \mid \nu_n \geq 0, \forall n \in \mathcal{S}_+ \text{ and } \nu_m \leq 0, \forall m \in \mathcal{S}_-\}$, where $\mathcal{S}_+ := \{n = 1, \ldots, N \mid u_n = \infty\}$ and $\mathcal{S}_- := \{m = 1, \ldots, N \mid l_m = -\infty\}$.

[6] $\mathbf{0}_N$ is the column vector of zeros. Note that the equality is componentwise equality.

[7] Please see Appendix A for further details.

[8] While constraints on short-selling and the incomplete market case (i.e. less than N risky assets) may be interesting, their application is more straightforward than the two cases considered in the study. In effect, the types of constraints we consider provide enough generalization; for example, we can recover short-selling prohibition, short-selling constraints, restrictions on long positions (not to be confused with borrowing constraints) from the rectangular constraints.

Next, for every process $\nu(\cdot) \in \mathcal{D}$, we express the assets of the auxiliary market by

$$\begin{aligned} \mathrm{d}V_0(t) &= V_0(t)(r + \delta(\nu(t)))\mathrm{d}t; & (2.6)\\ \mathrm{d}S_i^{\nu}(t) &= S_i^{\nu}(t)\left[(\mu_i + \nu_i(t) + \delta(\nu(t)))\mathrm{d}t + \sum_{j=1}^{N}\sigma_{ij}\mathrm{d}B_j(t)\right], \quad \text{for } i = 1, \ldots, N. & (2.7) \end{aligned}$$

Then, the wealth process is the solution of the stochastic differential equation

$$\mathrm{d}X_{\nu}^{\boldsymbol{w}}(t) = X_{\nu}^{\boldsymbol{w}}(t)\left[(r + \delta(\nu(t)))\mathrm{d}t + \boldsymbol{w}'(t)(\mu + \nu(t) - r\underset{\sim}{1})\mathrm{d}t + \boldsymbol{w}'(t)\sigma\mathrm{d}B^{\bullet}(t)\right]. \quad (2.8)$$

Given the wealth process dynamics in (2.8), the benchmarked portfolio process in the auxiliary market is the solution of

$$\begin{aligned} \mathrm{d}Z_{\nu}^{\boldsymbol{w}}(t) &= Z_{\nu}^{\boldsymbol{w}}(t)\left[\left(\hat{r} + \delta(\nu(t)) + \boldsymbol{w}'(t)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\right)\mathrm{d}t\right. \\ &\qquad \left. + (\boldsymbol{w}'(t)\sigma - b')\mathrm{d}B^{\bullet}(t) - \beta\mathrm{d}B^{(N+1)}(t)\right]. \quad (2.9) \end{aligned}$$

Moreover, we define

$$\begin{aligned} \zeta_{\nu}(t) &:= \sigma^{-1}(\hat{\mu} + \nu(t) - r\underset{\sim}{1}) \\ &= \zeta + \sigma^{-1}\nu(t), \end{aligned}$$

where $\zeta := \sigma^{-1}(\hat{\mu} - r\underset{\sim}{1})$. We also let $\tilde{\zeta} := \sigma^{-1}(\mu - r\underset{\sim}{1})$ denote the market price of risk.

Next, we define the second-order differential operator $\mathcal{L}_{\nu}^{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \mathbb{R}^N$ and $\nu \in \mathbb{R}^N$ as in the following way: For every open set $\mathcal{O} \subset \mathbb{R}$ and for every $\Upsilon_{\nu}(x) \in C^2(\mathcal{O})$ the function $\mathcal{L}_{\nu}^{\boldsymbol{w}}\Upsilon_{\nu} : \mathcal{O} \to \mathbb{R}$ is given by

$$\mathcal{L}_{\nu}^{\boldsymbol{w}}\Upsilon_{\nu}(z) := \left((\hat{r} + \delta(\nu))\mathrm{d}t + \boldsymbol{w}'(\hat{\mu} + \nu - r\underset{\sim}{1})\right) z\Upsilon_{\nu}'(z) + \frac{1}{2}\left(\|\boldsymbol{w}'\sigma - b'\|^2 + \beta^2\right) z^2\Upsilon_{\nu}''(z), \quad (2.10)$$

where $\Upsilon_{\nu}'(z)$ is the first derivative and $\Upsilon_{\nu}''(z)$ is the second derivative with respect to z. We use throughout the paper the parameters defined as $\Sigma := \sigma\sigma'$, $D := \zeta'\sigma^{-1}\underset{\sim}{1}$, $K := \underset{\sim}{1}'\Sigma^{-1}\underset{\sim}{1}$ and $Q := b'\sigma^{-1}\underset{\sim}{1}$. Finally, note that $\hat{r} + b'\zeta = r - \alpha + b'\tilde{\zeta} + \beta^2$ and that $\|\zeta\|^2 = \|\tilde{\zeta}\|^2 - 2b'\tilde{\zeta} + \|b\|^2$. These equalities are used interchangeably when necessary for reporting and/or interpreting the results.

3 Problem Formulation

In this section, we provide the generalized version of three problems we consider in this study. With this in mind, we let for $L < x < U$,

$$\tau_L^{\boldsymbol{w}} := \inf\{t > 0 | Z_{\nu}^{\boldsymbol{w}}(t) \leq L\},$$

be the first time the benchmarked auxiliary portfolio process $Z_{\nu}^{\boldsymbol{w}}(\cdot)$ crosses the lower level L, and

$$\tau_U^{\boldsymbol{w}} := \inf\{t > 0 | Z_{\nu}^{\boldsymbol{w}}(t) \geq U\},$$

be the first time the benchmarked auxiliary portfolio process $Z_\nu^{\boldsymbol{w}}(\cdot)$ crosses the upper level U. Using $\tau_L^{\boldsymbol{w}}$ and $\tau_U^{\boldsymbol{w}}$, we also define $\tau^{\boldsymbol{w}} := \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}$. Mainly, $\tau^{\boldsymbol{w}}$ denotes the first escape time of $Z_\nu^{\boldsymbol{w}}(\cdot)$ from the interval (L, U) under an admissible control process $\{\boldsymbol{w}(t), t \geq 0\}$.

Now we express the general form by

$$\begin{aligned} J_\nu^{\boldsymbol{w}}(z) \;&:=\; \mathbb{E}_z\Bigg[\int_0^{\tau^{w}} \exp\left\{-\int_0^s \rho(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s\right\} q(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s \\ &\qquad + \exp\left\{-\int_0^{\tau^{w}} \rho(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s\right\} H(Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}}))\Bigg], \end{aligned} \tag{3.1}$$

where $\rho(z) \geq 0$ is a real-valued function, $q(z)$ is a real-valued bounded and continuous function and $H(z)$ is defined on a domain set $z \in \{L, U\}$. To shorten the notation, we use $\mathbb{E}_z[\cdot] = \mathbb{E}[\cdot \mid Z(0) = z]$. Likewise, we set $\mathbb{P}_z(\cdot) = \mathbb{P}(\cdot \mid Z(0) = z)$.

Remark 3.1. *As shown in Browne (1999) (see also Browne (1997)), by setting $\rho(\cdot) = 0$, $q(\cdot) = 1$, and $H(L) = H(U) = 0$, we obtain from (3.1), the value function $\mathbb{E}_z\left[\tau^{\boldsymbol{w}}\right]$, which is the expected exit time from the interval (L, U). This yields the value functions of the first problem. On the other hand, if we set $\rho(\cdot) = q(\cdot) = 0$, $H(L) = 0$ and $H(U) = 1$, we obtain from (3.1), $\mathbb{P}_z\left(\tau_U^{\boldsymbol{w}} < \tau_L^{\boldsymbol{w}}\right)$, the value function for the maximization of the probability that the benchmarked wealth process hits U without first hitting L. Finally, for $q(\cdot) = 0$, $\rho(\cdot) = \rho > 0$ and $H(U) = 1$, the value function in (3.1) becomes $\mathbb{E}_z\left[e^{-\rho\tau_U^{\boldsymbol{w}}}\right]$, which is the value function for the expected discounted reward maximization problem. The expected discounted penalty minimization may also be obtained similarly.*

Depending on the problems we are solving in the study, the objective in turn involves either the maximization or minimization of (3.1). Thus,

$$\bar{V}_\nu(z) := \sup_{\boldsymbol{w} \in \mathcal{A}_\nu} J_\nu^{\boldsymbol{w}}(z) \quad \text{or} \quad \underline{V}_\nu(z) := \inf_{\boldsymbol{w} \in \mathcal{A}_\nu} J_\nu^{\boldsymbol{w}}(z), \tag{3.2}$$

where $\mathcal{A}_\nu(z)$ is the set of admissible strategies in the auxiliary market[9]. As mentioned in Section (2.3), when the constraints are binding we have $\delta(\nu) + \boldsymbol{w}'\nu \geq 0 \ \forall \nu \in \tilde{\mathcal{K}}$ if and only if $\boldsymbol{w} \in \mathcal{K}$. Therefore, once we find a fictitious parameter $\nu^* \in \tilde{\mathcal{K}}$ and establish that $\boldsymbol{w}^* \in \mathcal{K}$, then we obtain $\delta(\nu^*) + \boldsymbol{w}^{*\prime}\nu^* = 0$[10]. Furthermore, once the optimality is shown, the constrained values of the maximization and minimization problems are respectively given by,

$$\bar{V}_{\nu^*}(z) := \inf_{\nu \in \mathcal{D}} \bar{V}_\nu(z), \qquad \underline{V}_{\nu^*}(z) := \sup_{\nu \in \mathcal{D}} \underline{V}_\nu(z),$$

where $V_{\nu^*}(\cdot)$[11] is the optimal value function under the constrained markets.

Next, we provide the theorem that we will use to solve the problems considered in the study. With appropriate choices for $\rho(\cdot)$, $q(\cdot)$, and $H(\cdot)$ we may apply the theorem to solve the three problems. Notice that the theorem is for the specifications under the auxiliary market. The specifications under the constrained markets will be provided when solving the problems. We also note at the outset that we only provide the specifications under the constrained markets. The specifications under the unconstrained markets can be seen in Browne (1999).

[9]The admissibility in the auxiliary market is defined similarly. That is, we say that $\boldsymbol{w}(t) \in \mathcal{A}_\nu(z)$ if $\boldsymbol{w}(t)$ is $\{\mathcal{F}_t\}$-progressively measurable, satisfies $\int_0^t \|\boldsymbol{w}(s)\|^2 ds < \infty$ almost surely for $t < \infty$.

[10]Please see Appendix A for further details.

[11]The subscript ν^* is used to denote the values under the constrained markets.

Theorem 3.1. *Suppose $G_\nu : [L, U] \to \mathbb{R}$ is a concave increasing function, smooth enough in the sense $G_\nu \in C^2((L, U))$. Furthermore, assume for $c = (c_1, c_2, c_3)$, $c \geq 0$, that $|G_\nu(z)| < c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$ and that G_ν*[12] *satisfies the non-linear partial differential equation for $\nu \in \tilde{\mathcal{K}}$*

$$-\rho(z)G_\nu + q(z) + (\hat{r} + \delta(\nu) + b'\zeta_\nu)zG'_\nu - \frac{1}{2}\|\zeta_\nu\|^2 \frac{(G'_\nu)^2}{G''_\nu} + \frac{1}{2}\beta^2 z^2 G''_\nu = 0, \tag{3.3}$$

subject to the boundary conditions $G_\nu(L) = H(L)$, $G_\nu(U) = H(U)$. If $\boldsymbol{w}^*(x)$ is such that $-\rho(z)G_\nu(z) + q(z) + \mathcal{L}_\nu^{\boldsymbol{w}^*} G_\nu(z) = 0$ and $\boldsymbol{w}^*(Z_\nu^*(t))$ is admissible, then, $G_\nu(z)$ is the optimal value function under the auxiliary market (That is, $G_\nu(z) = V_\nu(z)$) and $\boldsymbol{w}^*(Z_\nu^*(t))$ is the optimal investment strategy, where $Z_\nu^*(t)$ is the optimal benchmarked wealth process for $t < \tau^w$. Consequently, the optimal investment strategy under the auxiliary market is

$$\boldsymbol{w}^*(z) = (\sigma^{-1})'b - (\sigma^{-1})'\zeta_\nu(z)\frac{G'_\nu}{zG''_\nu}. \tag{3.4}$$

Proof: Please see Appendix B.

4 Maximizing/Minimizing The Expected Time

In this section we will introduce the first problem. We consider an investor who is interested in minimizing (maximizing) the expected time to reach (stay above) an upper (a lower) wealth level with respect to a benchmark. When investing, the investor takes into consideration the condition of the markets so that she can decide whether her portfolio may attain the desired objective. To model the objective, we denote a condition with a market favorability parameter whose value depends on the direction of the trend of the investor's portfolio. In what follows, we will see that our investor can minimize the expected time until her portfolio reaches an upper level if and only if the markets are favorable. On the other hand, if the markets are unfavorable, she can only invest to maximize the expected time to stay above a predetermined wealth level. To clarify, we provide the following remark.

Remark 4.1. *For the specification of the favorability parameter under the auxiliary market, we first consider the strategy that maximizes the growth rate of the benchmarked portfolio $Z_\nu^{\boldsymbol{w}}(\cdot)$. We call this strategy the growth optimal strategy*[13]*. To specify, we first write, by the application of Itô's formula to $\log(Z_\nu^{\boldsymbol{w}}(t))$, the stochastic differential equation for $\boldsymbol{w} \in \mathbb{R}^N$ and $\nu \in \mathcal{K}$*

$$\begin{aligned} \mathrm{d}(\log(Z_\nu^w(t))) \quad = \quad & \left(\hat{r} + \delta(\nu) + \boldsymbol{w}'(\hat{\mu} + \nu - r\underset{\sim}{1}) - \frac{1}{2}\left(\|\boldsymbol{w}'\sigma - b'\|^2 + \beta^2\right)\right)\mathrm{d}t \\ & + (\boldsymbol{w}'\sigma - b')\mathrm{d}B^\bullet(t) - \beta \mathrm{d}B^{(N+1)}(t). \end{aligned} \tag{4.1}$$

The strategy that maximizes the growth rate can then be found by solving

$$\boldsymbol{w}_o^* := \arg\sup_{\boldsymbol{w}} \left[\hat{r} + \delta(\nu) + \boldsymbol{w}'(\hat{\mu} + \nu - r\underset{\sim}{1}) - \frac{1}{2}\left(\|\boldsymbol{w}'\sigma - b'\|^2 + \beta^2\right)\right].$$

A pointwise optimization yields the auxiliary growth optimal strategy $\boldsymbol{w}_o^* = (\sigma^{-1})'\tilde{\zeta}_\nu$, where $\tilde{\zeta}_\nu := \sigma^{-1}(\mu + \nu - r\underset{\sim}{1})$.

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[12] We hide the argumements of the functions when necessary to simplify the notation.

[13] We refer the readers to Maclean et al. (2010) for a through review of the properties of this strategy.

Remark 4.2. *In the light of Remark 4.1, substituting* $\boldsymbol{w}_o^* = (\sigma^{-1})'\tilde{\zeta}_\nu$ *to Equation (4.1) and solving for* $Z_\nu^{\boldsymbol{w}_o^*}(t)$ *by the application of Itô's formula yields for* $t < \infty$

$$Z_\nu^{\boldsymbol{w}_o^*}(t) = Z(0)\exp\{\theta_\nu t + \tilde{\zeta}_\nu' B^\bullet(t) - \beta B^{N+1}(t)\}, \tag{4.2}$$

where

$$\theta_\nu := r + \delta(\nu) - \alpha + \frac{1}{2}\left(\|\tilde{\zeta}_\nu\|^2 + \|b\|^2 + \beta^2\right). \tag{4.3}$$

We name θ_ν *as the favorability parameter under the auxiliary market. Under the unconstrained market, we have* $\nu = \boldsymbol{0}_N$ *and the favorability parameter becomes equal to the favorability parameter considered in Browne (1999). When* $\theta_\nu > 0$ *(favorable market), as* $t \to \infty$ *we have* $Z_\nu^{\boldsymbol{w}_o^*}(t) \to \infty$*, implying that* $\tau_U^{\boldsymbol{w}_o^*} < \infty$ $\mathbb{P}$*-almost surely. On the other hand, when* $\theta_\nu < 0$ *(unfavorable market),* $Z_\nu^{\boldsymbol{w}_o^*}(t) \to 0$ *as* $t \to \infty$*, implying this time that* $\tau_L^{\boldsymbol{w}_o^*} < \infty$ $\mathbb{P}$*-almost surely. Therefore, depending on the level of favorability, one of the hitting times becomes finite.*

As a consequence of Remark 4.2, we consider two different problems. These are namely the minimization of the expected time to beat a benchmark and the maximization of the expected time until shortfall. The mathematical formulation of the former and the latter problems are respectively given by

$$\bar{E}_\nu(z) := \inf_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[\tau_U^{\boldsymbol{w}}], \qquad \underline{E}_\nu(z) := \sup_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[\tau_L^{\boldsymbol{w}}].$$

We will see in the next theorem that two different problems have the same investment strategy.

Theorem 4.1. *Let the vector of optimal fictitious parameters be*[14]

$$\nu^* = \begin{cases} \left(\frac{\kappa-\tilde{D}}{K}\right)\underset{\sim}{1} & \text{if } \tilde{D} \geq \kappa; \\ \epsilon_1^* & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N, \end{cases} \tag{4.4}$$

where $\tilde{D} := \tilde{\zeta}'\sigma^{-1}\underset{\sim}{1}$ *and* $\epsilon_1^* := \arg\min_{\nu\in\tilde{\mathcal{K}}} \theta_\nu$*, with* θ_ν *is given in (4.3). The favorability parameter* $\theta_{\nu^*} \in \mathbb{R}\setminus\{0\}$ *is thereby*

$$\theta_{\nu^*} := \begin{cases} \theta - \frac{(\kappa-\tilde{D})^2}{2K} & \text{if } \tilde{D} \geq \kappa; \\ \theta + u'\epsilon_1^{*-} - l'\epsilon_1^{*+} + \tilde{\zeta}'\sigma^{-1}\epsilon_1^* + \frac{1}{2}\epsilon_1^{*\prime}\Sigma^{-1}\epsilon_1^* & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N, \end{cases} \tag{4.5}$$

with $\theta = r - \alpha + 0.5(\|\tilde{\zeta}\|^2 + \|b\|^2 + \beta^2)$*, the favorability parameter under the unconstrained market. Then, the optimal value function is:*

$$\begin{cases} \bar{E}_{\nu^*}(z) = \frac{1}{\theta_{\nu^*}}\log\left(\frac{U}{z}\right) & \text{for } z \leq U \text{ and } \theta_{\nu^*} > 0; \\ \underline{E}_{\nu^*}(z) = \frac{1}{|\theta_{\nu^*}|}\log\left(\frac{z}{L}\right) & \text{for } z \geq L \text{ and } \theta_{\nu^*} < 0. \end{cases} \tag{4.6}$$

[14] $(\sigma^{-1})'\tilde{\zeta}$ is the growth optimal strategy under the unconstrained case. From $\tilde{D} := \tilde{\zeta}'\sigma^{-1}\underset{\sim}{1}$, we use $\tilde{D} \geq \kappa$ to indicate the case of the borrowing constraints. For the rectangular constraints w_n^* is a component of the unconstrained strategy $(\sigma^{-1})'\tilde{\zeta}$. It may be possible that some or all of the components violate their respective constraints denoted by the intervals $[l_n, u_n]$ for $n = 1, \ldots, N$.

In both cases, the optimal investment strategy for $L < x < U$ is

$$\boldsymbol{w}^*(z) = \begin{cases} (\sigma^{-1})' \left(\tilde{\zeta} + \sigma^{-1} \left(\frac{\kappa - \tilde{D}}{K} \right) \underset{\sim}{1} \right) & \text{if } \tilde{D} \geq \kappa; \\ (\sigma^{-1})' \left(\tilde{\zeta} + \sigma^{-1} \epsilon_1^* \right) & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N. \end{cases} \tag{4.7}$$

Proof: Please see Appendix C. □

We observe that the results change based on the favorability level of the markets. This is also evident from the proof in Appendix C, since $\tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}} < \infty$ $\mathbb{P}-$almost surely for $\theta_{\nu^*} \neq 0$. In this case, when $\theta_{\nu^*} < 0$, we have $\sup_{\boldsymbol{w}} \mathbb{E}_z[\tau_L^{\boldsymbol{w}}] < \infty$ and when $\theta_{\nu^*} > 0$, $\inf_{\boldsymbol{w}} \mathbb{E}_z[\tau_U^{\boldsymbol{w}}] < \infty$. Therefore, for $\theta_{\nu^*} \neq 0$, the benchmarked portfolio process, $Z_\nu^{\boldsymbol{w}}(\cdot)$, crosses one of the boundary levels L or U $\mathbb{P}-$almost surely depending on the favorability of the markets.

In each case, as we see in specification (4.7), the optimal investment strategy is the growth optimal strategy. From its specification, we observe that the strategy is independent of the parameters of the benchmark process. That is, the values of α, b', and β parameters of the exogenous benchmark process do not have any effect on the growth policy[15]. Therefore, an investor maximizes growth as though she was investing without referencing to a benchmark. With the growth maximizing strategy, the investor hopes that the compounding effect brings success as quickly as possible if the markets are favorable or helps the survival of the portfolio above a certain level over the benchmark as long as possible if the markets are unfavorable.

However, notice that the growth rate θ_{ν^*} is dependent on the benchmark parameters α and b'. Therefore, the change in the return and risk characteristic of the benchmark has an effect on the favorability condition. Moreover, from (4.5), we may deduce the dependence of the favorability on the value of the constraint violation given by the optimal fictitious parameters. For example, under the borrowing constraints, $\theta_{\nu^*} > 0$ implies $\theta > (\kappa - \tilde{D})^2/2K$ showing that the favourability parameter of the unconstrained case must be larger than a positive factor of $(\kappa - \tilde{D})^2/2K$ under the constrained case so that the investor can find an optimal investment strategy to reach her goal[16]. This, in turn, implies that there may be some cases when a portfolio manager cannot afford to aim the upper goal while she could if borrowing was allowed. In other words, an investor needs to wait the markets to be favorable enough so that she may be able to beat the benchmark.

Next, we proceed to solve the remaining two problems. From the results, we will see the dependence of the optimal investment strategies on the parameters of the benchmark process.

5 Maximizing the Probability of Beating the Benchmark

In this section, we consider an investor who is interested in reaching a higher wealth level without first hitting a lower one. That is, the investor is interested in beating a benchmark before incurring a certain level shortfall with respect to it. Thus, we define the second problem as

$$P_\nu(z) := \sup_{\boldsymbol{w} \in \mathcal{A}_\nu} \mathbb{P}_z(\tau_U^{\boldsymbol{w}} > \tau_L^{\boldsymbol{w}}),$$

[15]Under the rectangular constraints, we may see from (C.5) that the optimal fictitious parameter value is not dependent on the parameters of the benchmark process as well.

[16]We only comment on the effect of borrowing constraints. The effect of rectangular constraints may be interpreted similarly. Therefore, it is hidden.

which is maximizing the probability of reaching a higher target before incurring a certain level of shortfall with respect to a benchmark.

Remark 5.1. *As shown in Browne (1999), under the unconstrained market, when* $\beta = 0$*, an investor may set* $\boldsymbol{w} = (\sigma^{-1})'b$ *(which is the minimum variance strategy) to eliminate the controllable risk of the benchmarked portfolio process (see the terms before* $\mathrm{d}B^{\bullet}(t)$ *in (2.5)). In this case, the closed form solution of the benchmarked portfolio process becomes*

$$Z^{\boldsymbol{w}}(t) = z \exp\left\{(r - \alpha + b'\tilde{\zeta})t\right\}.$$

Then, if $r + b'\tilde{\zeta} \geq \alpha$[17] *an investor will never incur a shortfall with respect to a benchmark, implying a riskless growth for the ratio process. Therefore, the interesting case when* $\beta = 0$ *requires* $r + b'\tilde{\zeta} < \alpha$[18] *so that the investor cannot eliminate all risks by pursuing the minimum variance strategy. As we will see, when* $\beta = 0$*, the condition must also hold in some cases under the constrained markets for the problem to have an optimal solution.*

We provide the results of this problem in the theorem that follows.

Theorem 5.1. *Let the vector of optimal fictitious parameters be*[19]

$$\nu^* = \begin{cases} \left(\frac{\gamma^-}{K}(-\kappa + Q) - \frac{D}{K}\right)\underset{\sim}{1} & \text{if } Q - \frac{1}{\gamma^-}D \geq \kappa; \\ \epsilon_2^* & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N, \end{cases} \tag{5.1}$$

where $\epsilon_2^* := \arg\sup_{\nu \in \tilde{\mathfrak{K}}} \gamma^-(\nu)$ *with* $\gamma^-(\nu)$ *defined in (D.7) and* $\gamma^- \in (-\infty, 0) \setminus \{-1\}$ *for* $\|\zeta\|^2 - \frac{D^2}{K} \geq 0$ *is*

$$\gamma^- = \begin{cases} -\frac{\left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa+Q)\right) + \sqrt{\Xi}}{\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right)} & \text{if } Q - \frac{1}{\gamma^-}D \geq \kappa; \\ -\frac{\left(\hat{r} + u'\epsilon_2^{*-} - l'\epsilon_2^{*+} + b'\zeta_{\epsilon_2^*}\right) + \sqrt{\Xi}}{\beta^2} & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N, \end{cases} \tag{5.2}$$

with $\zeta_{\epsilon_2^*} = \zeta + \sigma^{-1}\epsilon_2^*$*, and* Ξ *is*

$$\Xi = \begin{cases} \left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa + Q)\right)^2 + \left(\frac{1}{K}(-\kappa + Q)^2 + \beta^2\right)\left(\|\zeta\|^2 - \frac{D^2}{K}\right) & \text{if } Q - \frac{1}{\gamma^-}D \geq \kappa; \\ \left(\hat{r} + u'\epsilon_2^{*-} - l'\epsilon_2^{*+} + b'\zeta_{\epsilon_2^*}\right)^2 + \beta^2\|\zeta_{\epsilon_2^*}\|^2 & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N. \end{cases} \tag{5.3}$$

[17] The inequality implies that the benchmark's growth rate is less then the rate of growth of the minimum variance portfolio.

[18] Such condition does not need to hold when $\beta \neq 0$, since there is always an uncontrollable source of randomness that may lead to a shortfall.

[19] The optimal investment strategy for the second problem under the unconstrained case is $(\sigma^{-1})'b - \frac{1}{\gamma^-}(\sigma^{-1})'\zeta$, where γ^- is also specified under the unconstrained case. By this specification, we let $Q - \frac{1}{\gamma^-}D \geq \kappa$ denote the case when the borrowing constraints bind since $\underset{\sim}{1}'[(\sigma^{-1})'b - \frac{1}{\gamma^-}(\sigma^{-1})'\zeta] \geq \kappa \Rightarrow Q - \frac{1}{\gamma^-}D \geq \kappa$. The specification of the case when the rectangular constraints bind follows similarly from the previous problem.

Then, the optimal value function, $P_{\nu^*}(z) \in [0,1]$, is given by

$$P_{\nu^*}(z) = \frac{L^{1+\gamma^-} - z^{1+\gamma^-}}{L^{1+\gamma^-} - U^{1+\gamma^-}}, \tag{5.4}$$

and the optimal investment strategy is for $L < x < U$

$$\boldsymbol{w}^*(z) = \begin{cases} (\sigma^{-1})'b - \frac{1}{\gamma^-}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\left(\frac{\gamma^-}{K}(-\kappa + Q) - \frac{D}{K}\right)\mathbb{1}\right) & \text{if } Q - \frac{1}{\gamma^-}D \geq \kappa; \\ (\sigma^{-1})'b - \frac{1}{\gamma^-(\epsilon_2^*)}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\epsilon_2^*\right) & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N. \end{cases} \tag{5.5}$$

Proof: Please see Appendix D. □

In this problem, we observe the dependence of the optimal investment strategy to the parameters of the benchmark process. This is explicit from $(\sigma^{-1})'b$, which is the minimum variance strategy and γ^- in (5.5). The term γ^- is interesting as its value is related to the strength of the markets. That is, applying the arguments in Browne (1999) to our case, we may observe that

$$\gamma^- \begin{cases} < -1 & \text{if } \theta_{\nu^*} > 0; \\ = -1 & \text{if } \theta_{\nu^*} = 0; \\ > -1 & \text{if } \theta_{\nu^*} < 0. \end{cases} \tag{5.6}$$

Note that the optimal strategy under the unconstrained markets can be directly recovered by setting the value of the optimal fictitious parameters to zero. Here, we provide the interpretation of the results under the unconstrained market case as parts of the interpretation under the constrained case follow similarly[20]. More details regarding the interpretation of the results under the constrained case are provided in the numerical analysis section (see Section 7).

Then, it follows from (5.6) that, under the unconstrained case, as markets become favorable, $\gamma^- \downarrow -\infty$ and $\boldsymbol{w}^*(z) \to (\sigma^{-1})'b$. On the other hand, for the same case, as markets become unfavorable, $\gamma^- \uparrow 0$ and $\boldsymbol{w}^*(z) \uparrow \infty$. Therefore, with deteriorating market conditions, an investor, who is not facing any investment constraints, may invest more aggressively to maximize the probability of success with respect to a benchmark. The risk level taken may in fact be higher than that taken under the growth optimal strategy[21]. That is, the strategy followed for maximizing the probability of success may be bolder than the growth optimal strategy as markets become more unfavorable. However, for values of γ^- close to -1 under the unfavorable markets, the strategy may not necessarily be bolder. This follows from the third example in Section 7.

On the other hand, as the conditions of the markets ameliorate, the success probability maximizing strategy becomes closer to the minimum variance strategy $(\sigma^{-1})'b$. The risk level taken is therefore more associated with the minimum variance strategy, the better the conditions of the markets are. This association has implications in terms of the risk level taken. If the components of $(\sigma^{-1})'b$ are large, then, the investment exposure under the success probability maximizing strategy may not necessarily be less than that taken under the growth optimal strategy. Extreme values of b' may yield very high values for $(\sigma^{-1})'b$, which might in turn lead to bolder values than the growth optimal strategy. As a result, favorability of the markets do

[20]This is mainly true for the borrowing constraints.

[21]Note that for $\gamma^- = -1$ the optimal investment strategy becomes the growth optimal strategy.

not necessarily imply timid strategy under the maximization of the probability of successwith respect to a benchmark[22].

5.1 What Happens when $\beta = 0$?

Under the unconstrained markets, when $\beta = 0$, we obtain $\gamma^- = 0.5\|\zeta\|^2/(r-\alpha+b'\tilde{\zeta})$. Since the problem has a unique optimal solution for $\gamma^- < 0$, then the condition $r-\alpha+b'\tilde{\zeta} < 0$ must hold. Otherwise, Remark 5.1 implies that an investor could have solely pursued the minimum variance strategy $(\sigma^{-1})'b$ to guarantee an absolute gain, or at least, to prevent a shortfall against a benchmark if $r-\alpha+b'\tilde{\zeta} \geq 0$. Therefore, interesting case for the maximization of the probability of success with respect to a benchmark happens when $r-\alpha+b'\tilde{\zeta} < 0$.

However, whether $r-\alpha+b'\tilde{\zeta} < 0$ holds or not under the unconstrained markets depends on the parametrization of α and b'. For example, we could have set for a fixed proportional investment strategy $\acute{\boldsymbol{f}} = (f_1, \ldots, f_N)'$, $\alpha = r + \acute{\boldsymbol{f}}(\mu - r\underset{\sim}{1})$ and $b' = \acute{\boldsymbol{f}}\sigma$. In this way, the benchmark process becomes a benchmark portfolio process traded with a fixed proportional strategy $\boldsymbol{f}$. Substitution of the new parametrized values of α and b' gives in turn $r-\alpha+b'\tilde{\zeta} = 0$.

We have $r-\alpha+b'\tilde{\zeta} = 0$, because an investor pursuing the minimum variance strategy $(\sigma^{-1})'b$, has $\boldsymbol{w}^* = (\sigma^{-1})'b = \boldsymbol{f}$ for $b' = \acute{\boldsymbol{f}}\sigma$. That is, an investor perfectly replicates the benchmark strategy so as the never incur a shortfall against it. Obviously, an investor perfectly replicating a benchmark portfolio must not be compensated with extra gain as this would suggest an arbitrage opportunity. As a result, maximizing the probability of success with respect to a benchmark portfolio is not possible under the complete markets because an investor may prevent shortfall by replicating the benchmark strategy. It becomes possible when the markets are incomplete due to the existence of an uncontrollable source of risk, non-traded asset, or as we will see next, due to investment constraints. Here, we note that when the uncontrollable source of risk exists (i.e. $\beta \neq 0$) and market conditions become stronger, if the benchmark is a portfolio process with strategy $\boldsymbol{f}$, then $\boldsymbol{w}^* \to \boldsymbol{f}$. So, the pursuit of a closer strategy to that of a benchmark is better for the maximization of the probability of success as the markets become favorable.

Finally, given the arguments above, we may deduce for the first problem that, when $\beta = 0$ and $r-\alpha+b'\tilde{\zeta} \geq 0$ the markets are always favorable (i.e. $\theta > 0$) under the unconstrained markets, since $\theta = r-\alpha+0.5(\|\tilde{\zeta}\|^2+\|b\|^2) > r-\alpha+b'\tilde{\zeta} \geq 0$[23] provided that $\tilde{\zeta} \neq b$. Thus, the survival time maximization problem is not possible if $r-\alpha+b'\tilde{\zeta} \geq 0$ since there is no risk of shortfall. On the other hand, for $r-\alpha+b'\tilde{\zeta} \geq 0$, minimizing the time to beat a benchmark is still possible. However, an investor needs to take risk (i.e. growth optimal strategy) in order to fulfil the objective as the minimum variance strategy is not optimal for that problem.

5.2 Effects of Constraints when $\beta = 0$

We start the analysis under the borrowing constraints. We consider three cases: $\kappa = Q$, $\kappa > Q$ and $\kappa < Q$. We remind that $Q := b'\sigma^{-1}\underset{\sim}{1}$ is the sum of the minimum variance strategies.

[22]Though favorability does not necessarily imply timid strategy, there may be instances when when success probability maximizing strategy may be more timid than the growth optimal strategy. We show this with an example in the numerical analysis section. We also show the contrary in the same section.

[23]Observe that since $\|\tilde{\zeta}-b\|^2 > 0$ for $\tilde{\zeta} \neq b$, then $0.5(\|\tilde{\zeta}\|^2+\|b\|^2) > b'\tilde{\zeta}$ and $\theta > r-\alpha+b'\tilde{\zeta}$ follows.

We will see that the existence of result for the problem we consider in this section depends on the sign of $r - \alpha + b'\tilde{\zeta}$ when $\kappa \geq Q$.

From the first line value of (5.2) in Theorem 5.1, we have, for $\beta = 0$ and $\kappa = Q$, $\gamma^- = 0.5(\|\zeta\|^2 - D^2/K)/(r - \alpha + b'\tilde{\zeta})$. Then, for $\|\zeta\|^2 > D^2/K$, $\gamma^- < 0$ if $r - \alpha + b'\tilde{\zeta} < 0$. As we explained in the previous section, there will be a violation of the inequality $r - \alpha + b'\tilde{\zeta} < 0$ if the benchmark is a portfolio process. It follows that when $\kappa = Q$, then the problem considered in this section has a unique solution under the borrowing constraints only if $r - \alpha + b'\tilde{\zeta}$ remains negative (which is possible if the benchmark is not a portfolio process).

When $\kappa > Q$, an investor may always have the chance to set her trading strategy equal to the minimum variance strategy $(\sigma^{-1})'b$ to eliminate all risks. In this case as well, we must have $r - \alpha + b'\tilde{\zeta} < 0$, which, again, is possible if the benchmark is not a portfolio process. Finally, if $\kappa < Q$, regardless of the type of benchmark and the sign of $r - \alpha + b'\tilde{\zeta}$, the problem will always have a unique solution. This is because, an investor cannot pursue the minimum variance strategy $(\sigma^{-1})'b$ when $\kappa < Q$. As a result, there will always be a certain level of risk that must be taken to maximize the probability problem considered in this section.

In sum, when the uncontrollable source of risk is absent and there are borrowing constraints, maximizing the probability of success with respect to a benchmark is possible under two cases: (i) Either the borrowing constraint level must be set low enough to prevent the pursuit of the minimum variance investment strategy $(\sigma^{-1})'b$, or (ii) The benchmark's growth rate must be higher than the rate of growth of the minimum variance portfolio. The latter is not possible if the benchmark is a portfolio process.

Under the rectangular constraints, when $\beta = 0$, γ^- becomes for $\epsilon_2^* \in \tilde{\mathcal{K}}$

$$\gamma^- = \frac{\|\zeta_{\epsilon_2^*}\|^2}{r + u'\epsilon_2^{*-} - l'\epsilon_2^{*+} - \alpha + b'\tilde{\zeta}_{\epsilon_2^*}}.$$

Here, the denominator must be in the form $r - \alpha + u'\epsilon_2^{*+} - l'\epsilon_2^{*-} + b'\tilde{\zeta}_{\epsilon_2^*} < 0$ so that we can have $\gamma^- < 0$ and the problem may have a unique solution. Notice from Appendix A that for the optimal fictitious parameter value $\epsilon_2^* \in \tilde{\mathcal{K}}$ and the optimal investment strategy $\boldsymbol{w}^* \in \mathcal{K}$, we have $\delta(\epsilon_2^*) + \boldsymbol{w}^{*'}\epsilon_2^* = 0$. Since, $\delta(\epsilon_2^*) = u'\epsilon_2^{*+} - l'\epsilon_2^{*-}$ and $r - \alpha + \delta(\epsilon_2^*) + b'\tilde{\zeta}_{\epsilon_2^*} < 0 = \delta(\epsilon_2^*) + \boldsymbol{w}^{*'}\epsilon_2^*$, we can write after rearranging the terms $r - \alpha + b'\tilde{\zeta} < \epsilon_2^{*'}(\boldsymbol{w}^* - (\sigma^{-1})'b)$. This inequality must hold so that the problem can have a unique solution under the rectangular constraints when $\beta = 0$. We note here as well that when $\boldsymbol{w}^* = (\sigma^{-1})'b$, the condition $r - \alpha + b'\tilde{\zeta} < 0$ must also hold.

6 Expected Discounted Reward/Penalty

In the first two sections we solved problems related to minimizing (maximizing) the time to beat (to stay above) a benchmark and maximizing the probability of beating a benchmark. In this section, we will consider the expected discounted reward and penalty problems. To this end, we define two different value functions. As in the first problem, we will see that the choice of the value function depends on the favorability of the market. More clearly, the investor may choose to maximize the expected discounted reward (goal reaching problem) or minimize the expected discounted penalty from a shortfall (survival problem) depending on the strength of the direction of the markets. For the reward and penalty problems, we write respectively

$$\bar{R}_\nu(z) := \sup_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[e^{-\rho\tau_U}], \qquad \underline{R}_\nu(z) := \inf_{\boldsymbol{w}\in\mathcal{A}_\nu} \mathbb{E}_z[e^{-\rho\tau_L}],$$

where $\rho > 0$ is the subjective constant discount rate. We present the solutions of the above problems in the theorems that follow. We first consider the maximization problem, then provide the results of the minimization problem.

Theorem 6.1. *Let $\hat{\eta}_b$ be one of the roots of (E.3) and $\hat{\eta}_b(\nu)$ be the solution of (E.7). Let also $\epsilon_3^{b*} := \arg\inf_{\nu\in\tilde{\mathcal{K}}} \hat{\eta}_b(\nu)$. If we have $-1 < \hat{\eta}_b, \hat{\eta}_b(\epsilon_3^{b*}) < 0$, and the portfolio is traded in a favorable market in the sense that $\theta_{\nu^*} > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_b}\right)^2 \left(\|\zeta\|^2 - \frac{D^2}{K}\right)$ under the borrowing and $\theta_{\nu^*} > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_b(\epsilon_3^{b*})}\right)^2 \|\zeta_{\nu^*}\|^2$ under the rectangular constraints, then the vector of optimal fictitious parameters that are substituted in θ_{ν^*} in (4.5) are*[24]

$$\nu^* = \begin{cases} \left(\frac{\hat{\eta}_b}{K}(-\kappa+Q) - \frac{D}{K}\right)\underset{\sim}{1} & \text{if } Q - \frac{1}{\hat{\eta}_b}D \ge \kappa; \\ \epsilon_3^{b*} & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \dots, N. \end{cases} \tag{6.1}$$

The optimal value function $\bar{R}_{\nu^*}(z) \in [0,1]$ is given by

$$\bar{R}_{\nu^*}(z) = \left(\frac{z}{U}\right)^{\hat{\eta}_b+1} \quad \text{for } z < U, \tag{6.2}$$

and the optimal investment strategy for $z < U$ is

$$\boldsymbol{w}^*(z) = \begin{cases} (\sigma^{-1})'b - \frac{1}{\hat{\eta}_b}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\left(\frac{\hat{\eta}_b}{K}(-\kappa+Q) - \frac{D}{K}\right)\underset{\sim}{1}\right) & \text{if } Q - \frac{1}{\hat{\eta}_b}D \ge \kappa; \\ (\sigma^{-1})'b - \frac{1}{\hat{\eta}_b(\epsilon_3^{b*})}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\epsilon_3^{b*}\right) & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \dots, N. \end{cases} \tag{6.3}$$

The proof of the problem is given in Appendix E. We observe from the results above that the markets must be favorable enough in the sense $\theta_{\nu^*} > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_b}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)$ under the borrowing and $\theta_{\nu^*} > \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_b}\right)^2\|\zeta_{\nu^*}\|^2$ under the rectangular constraints, so that an investor can maximize the value of her reward. Only under the two inequalities implying favorability that we have $\tau_U^{\boldsymbol{w}} < \infty$ $\mathbb{P}$-almost surely (see the proof in Appendix E). Otherwise, the solution to the maximization problem would not exist. We will see next that the unfavorable market leads to penalty minimization. In that case, we have $\tau_L^{\boldsymbol{w}} < \infty$ $\mathbb{P}$-almost surely. The interpretations for the effect of $\beta = 0$ will be provided in Section 6.1.

Theorem 6.2. *Let $\hat{\eta}_a$ be one of the roots of (E.3) and $\hat{\eta}_a(\nu)$ be the solution of (E.7). Let also $\epsilon_3^{a*} := \arg\sup_{\nu\in\tilde{\mathcal{K}}} \hat{\eta}_a(\nu)$. If we have $\hat{\eta}_a, \hat{\eta}_a(\epsilon_3^{a*}) < -1$, and the portfolio is traded in an*

[24]The optimal investment strategy for this problem under the unconstrained case is $(\sigma^{-1})'b - \frac{1}{\hat{\eta}_b}(\sigma^{-1})'\zeta$, where $\hat{\eta}_b$ is also specified under the unconstrained case (note that for the minimization problem $\hat{\eta}_b$ is replaced by $\hat{\eta}_a$). By the specification of the unconstrained investment strategy, we let $Q - \frac{1}{\hat{\eta}_b}D \ge \kappa$ denote the case when the borrowing constraints bind since $\underset{\sim}{1}'[(\sigma^{-1})'b - \frac{1}{\hat{\eta}_b}(\sigma^{-1})'\zeta] \ge \kappa \Rightarrow Q - \frac{1}{\hat{\eta}_b}D \ge \kappa$. The case of the rectangular constraints is expressed similar to our expression in the previous problems.

unfavorable market in the sense that $\theta_{\nu^*} < \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_a}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right)$ *under the borrowing and* $\theta_{\nu^*} < \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_a(\epsilon_3^{a*})}\right)^2\|\zeta_{\nu^*}\|^2$ *under the rectangular constraints, then the vector of optimal fictitious parameters that are substituted in* θ_{ν^*} *in (4.5) are*

$$\nu^* = \begin{cases} \left(\frac{\hat{\eta}_a}{K}(-\kappa+Q) - \frac{D}{K}\right)\mathbb{1} & \text{if } Q - \frac{1}{\hat{\eta}_a}D \geq \kappa; \\ \epsilon_3^{a*} & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N. \end{cases} \tag{6.4}$$

The optimal value function $\bar{R}_{\nu^*}(z) \in [0,1]$ *is then given by*

$$\underline{R}_{\nu^*}(z) = \left(\frac{L}{z}\right)^{\hat{\eta}_a+1} \quad \text{for } L < z, \tag{6.5}$$

and the optimal investment strategy for $L < z$ *is*

$$\boldsymbol{w}^*(z) = \begin{cases} (\sigma^{-1})'b - \frac{1}{\hat{\eta}_a}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\left(\frac{\hat{\eta}_a}{K}(-\kappa+Q) - \frac{D}{K}\right)\mathbb{1}\right) & \text{if } Q - \frac{1}{\hat{\eta}_a}D \geq \kappa; \\ (\sigma^{-1})'b - \frac{1}{\hat{\eta}_a(\epsilon_3^{a*})}(\sigma^{-1})'\left(\zeta + \sigma^{-1}\epsilon_3^{a*}\right) & \text{if } w_n^* \notin [l_n, u_n] \text{ for some} \\ & \text{or all } n = 1, \ldots, N. \end{cases} \tag{6.6}$$

Proof Please see Appendix E. □

We first interpret the results under the unconstrained markets. As we mentioned previously, the optimal investment strategies under the unconstrained markets can be obtained by setting the value of the optimal fictitious parameters to zero. Parts of the interpretation under the constrained case follow from the interpretation of the unconstrained case[25]. We provide further details regarding the analysis of the constrained case in Section 7.

Here as well, we observe the dependence of the optimal investment strategies of both problems to the parameters of the benchmark. We also see that, for $\hat{\eta}_. = -1$ (we set $\gamma^- = -1$ in the previous problem), the optimal investment strategies becomes equal to the growth optimal strategy. The difference of $\hat{\eta}_.$ from -1 makes the strategies dependent on the benchmark parameters. With market conditions deteriorating, we have $\hat{\eta}_a \downarrow -\infty$ and $\boldsymbol{w}^*(z) \to (\sigma^{-1})'b$. That is, as the market conditions become unfavorable, an investor will *eventually* consider close to minimum variance portfolio strategy when minimizing the expected discounted penalty from a shortfall. If the benchmark is a portfolio process traded with a proportional strategy $\boldsymbol{f}$ (see our explanation in Section 5.1), then setting $b' = \boldsymbol{f}'\sigma$ gives $\boldsymbol{w}^*(z) \to \boldsymbol{f}$ as $\hat{\eta}_a \downarrow -\infty$. In sum, when the benchmark is a portfolio process, an investor tends to pursue a strategy that is close to the strategy of the benchmark portfolio when minimizing the expected discounted penalty from a shortfall. Note that the same was true for the second problem under the *favorable* markets.

The question is whether the situation is reversed when the conditions are favorable. In essence, under the favorable and unconstrained markets, we have as $\hat{\eta}_b \uparrow 0$, $\boldsymbol{w}^*(z) \uparrow \infty$. Therefore, with improving market conditions, an investor not facing any investment constraints will *eventually* have a higher exposure to risk than an investor pursuing the growth optimal investment strategy. In contrast to the results of the previous problem, bolder strategy is pursued to maximize the expected discounted reward as the markets become *more and more* favorable. We note here as well that for $\hat{\eta}_b$ close to -1 the strategy may not necessarily be bolder given

[25] As we noted in the second problem, the statement is mainly true for the borrowing constraints.

its dependence on the benchmark parameters. We also show this with an example in Section 7. On the other hand, similar to our interpretation for the second problem, when the conditions deteriorate, an investor may not necessarily follow timid strategy because the investment strategy is limited by the minimum variance strategy as $\hat{\eta}_a \downarrow -\infty$.

6.1 Effects of $\beta = 0$ and The Constraints

Under the unconstrained markets when $\beta = 0$, the roots of $\hat{\eta}_{.}$ are given by[26]

$$\hat{\eta}^{\pm} = \frac{-\left(\hat{r} + b'\zeta - \frac{1}{2}\|\zeta\|^2 - \rho\right) \pm \sqrt{\left(\hat{r} + b'\zeta - \frac{1}{2}\|\zeta\|^2 - \rho\right)^2 + 2\|\zeta\|^2(\hat{r} + b'\zeta)}}{2(\hat{r} + b'\zeta)}.$$

We remind that $\hat{r} + b'\zeta = r - \alpha + b'\tilde{\zeta}$ and $\hat{\eta}^+\hat{\eta}^- > 0$ only for $r - \alpha + b'\tilde{\zeta} < 0$ provided that the discriminant is positive. In this case, both roots have the same sign and both are negative since $\hat{\eta}^+ + \hat{\eta}^- < 0$ with $\hat{\eta}^+ < -1 < \hat{\eta}^- < 0$. As a result, each problem has a unique solution.

For $r - \alpha + b'\tilde{\zeta} = 0$, we have one root of the form $\hat{\eta}^- = -0.5\|\zeta\|^2/(0.5\|\zeta\|^2 + \rho)$ with $-1 < \hat{\eta}^- < 0$. Furthermore, for $r - \alpha + b'\tilde{\zeta} > 0$, the only relevant root is again $\hat{\eta}^-$ and we have $-1 < \hat{\eta}^- < 0$. Therefore, when $r - \alpha + b'\tilde{\zeta} \geq 0$, only the reward maximizing strategy has a unique solution, since an investor can always prevent penalty by simply pursuing the minimum variance strategy $\boldsymbol{w}^* = (\sigma^{-1})'b$. However, pursuing the minimum variance strategy for reward maximization is not optimal, so the investor has to take risk to fulfil the objective. In this regard, the problem still has a unique solution.

The above result together with the results we found in Section 5.1 imply that, under the unconstrained markets, when $\beta = 0$ and $r - \alpha + b'\tilde{\zeta} \geq 0$ shortfall with respect to benchmark is not a possibility as risks can be eliminated by pursuing the minimum variance strategy $\boldsymbol{w}^* = (\sigma^{-1})'b$. However, to reach a goal, risk must be taken as designated in the goal reaching time minimization or reward maximization problems, because $(\sigma^{-1})'b$ is not optimal for these problems.

For the borrowing constraints, we consider the cases $\kappa \geq Q$ with $r - \alpha + b'\tilde{\zeta} \geq 0$ and $\kappa < Q$. The relevant value of $\hat{\eta}$ for each case can be found by manipulating (E.2) in Appendix E. Note that when $\kappa = Q$, (E.2) becomes a quadratic equation. On the other hand, for $\kappa \neq Q$, the roots are still the result of the cubic equation specified in (E.2). The existence of results are in turn determined by the sign of $r - \alpha + b'\tilde{\zeta}$. Here, findings for the cases related to $\kappa \geq Q$ are parallel to those of the unconstrained case. We do not repeat them again. When $\kappa < Q$ the sign of $r - \alpha + b'\tilde{\zeta}$ does not matter as the investor will not have the opportunity to pursue the minimum variance strategy $(\sigma^{-1})'b$ and eliminate the risk of shortfall.

In sum, when $\beta = 0$, from the results of the second and third problems, we see that risk minimization is possible if the borrowing limit set to a limit lower than the level that would allow an investor to eliminate the controllable sources of risk. In that instance, there is always a certain level of risk that the investor needs to take into account. On the other hand, if the borrowing limit is set high enough for an investor to eliminate the controllable source of risk, then, that investor's net return from such strategy must be negative. More clearly, an investor cannot eliminate the risk and ensure beating the benchmark at the same time.

[26] The quadratic equation that the roots solve can be recovered from (E.7) in Appendix E by setting $\beta = \delta(\nu) = 0$, $\nu = \mathbf{0}_N$. We omit further details as they follow from our solutions in the proof of this problem. Note that instead of using the notation $\hat{\eta}_a$ and $\hat{\eta}_b$ we use $\hat{\eta}^{\pm}$ to denote the roots.

For the rectangular constraints, we obtain for[27] $\epsilon_3^{*} \in \tilde{\mathcal{K}}$ and $\delta(\epsilon_3^{*}) = u^{'}\epsilon_3^{*-} - l^{'}\epsilon_3^{*+}$, when $\beta = 0$,

$$\hat{\eta}^{\pm} = \frac{-\left(\hat{r} + b^{'}\zeta + \delta(\epsilon_3^{*}) + b^{'}\sigma^{-1}\epsilon_3^{*} - \rho - \frac{1}{2}\|\zeta_{\epsilon_3^{*}}\|^2\right) \pm \sqrt{\chi}}{2(\hat{r} + b^{'}\zeta + \delta(\epsilon_3^{*}) + b^{'}\sigma^{-1}\epsilon_3^{*})},$$

with

$$\begin{aligned}\chi &= \left(\hat{r} + b^{'}\zeta + \delta(\epsilon_3^{*}) + b^{'}\sigma^{-1}\epsilon_3^{*} - \rho - \frac{1}{2}\|\zeta_{\epsilon_3^{*}}\|^2\right)^2 \\ &\qquad + 2\|\zeta_{\epsilon_3^{*}}\|^2(\hat{r} + b^{'}\zeta + \delta(\epsilon_3^{*}) + b^{'}\sigma^{-1}\epsilon_3^{*}).\end{aligned}$$

where $\zeta_{\epsilon_3^{*}} = \zeta + \sigma^{-1}\epsilon_3^{*}$. Assuming that the discriminant χ is positive, two roots are negative if $\hat{r} + b^{'}\zeta + \delta(\epsilon_3^{*}) + b^{'}\sigma^{-1}\epsilon_3^{*} < 0$. In addition, from $\delta(\epsilon_3^{*}) + \boldsymbol{w}^{*'}\epsilon_3^{*} = 0$, we may deduce that the two roots are negative when $r - \alpha + b'\tilde{\zeta} < \epsilon_3^{*'}(\boldsymbol{w}^{*} - (\sigma^{-1})'b)$. If $r - \alpha + b'\tilde{\zeta} > \epsilon_3^{*'}(\boldsymbol{w}^{*} - (\sigma^{-1})'b)$ we would only have a single negative root and that would only pertain to the reward maximization problem (as it did under the unconstrained markets). Therefore, the existence of results under the rectangular constraints when $\beta = 0$ depends on level of $r - \alpha + b'\tilde{\zeta}$ with respect to the threshold determined by the product of constraint violation (given by ϵ_3^{*}) and the difference between the optimal strategy and the minimum variance strategy.

7 Numerical Analysis

In this section, we provide the numerical analysis of our results by considering three examples. In the first example, the benchmark is a portfolio process with an uncontrollable source of risk. We let the benchmark portfolio weights be randomly chosen. We assume that the investment weights in the benchmark portfolio are long only and that their sum is equal to 1. In the second example, the benchmark is an equally weighted portfolio. Finally, in the third one, we analyse how large values for α and $b^{'}$ (i.e. benchmark process' return and risk levels) change the results. In the first example, we report all the values for the parameters of interest. For the second and third examples, only the relevant ones are reported. We note that some of the values might not precisely add up due to rounding.

The analysis mainly helps us to understand how the optimal investment strategies of the second and third problems change with respect to the growth optimal strategy given a certain set of values for the key parameters. The comparison is important, because the growth optimal strategy has the potential of causing large losses due to its low Arrow-Pratt risk aversion measure (see MacLean et al. (2011)). Therefore, comparison of an investment strategy with respect to the growth optimal strategy provides a view on the level of risk taken by an investor fulfilling an objective.

7.1 A First Example

[27] Here, we may have either ϵ_3^{a*} or ϵ_3^{b*} depending on the problem we are solving. For generalization we use ϵ_3^{*}.

We consider five stocks and set $r = 0.04$, $\rho = 0.03$, $\mu = [0.20, 0.15, 0.04, 0.08, 0.11]'$, $\alpha = 0.11066$, $b = [0.239186, 0.288597, 0.328357, 0.239685, 0.265710]'$, $\beta = 0.12$, and

$$\sigma = \begin{Vmatrix} 0.19 & 0.37 & 0.18 & 0.26 & 0.10 \\ 0.37 & 0.11 & 0.42 & 0.21 & 0.51 \\ 0.18 & 0.42 & 0.38 & 0.31 & 0.28 \\ 0.26 & 0.21 & 0.31 & 0.29 & 0.12 \\ 0.10 & 0.51 & 0.28 & 0.12 & 0.26 \end{Vmatrix}.$$

The parameters α and b' are chosen to match respectively the average rate of return and the volatility (pertaining to the controllable sources of risk) of a benchmark portfolio process traded with some proportional investment strategy[28] $\boldsymbol{f}'$. In this respect, we have $\alpha = r + \boldsymbol{f}'(\mu - r\underset{\sim}{1})$ and $b = \boldsymbol{f}'\sigma$. Here, the uncontrollable source of risk might represent for example the idiosyncratic risk that is associated with the management of the benchmark portfolio. We consider the case of a long only benchmark portfolio whose proportional investment weights are randomly chosen and their sum is equal to 1. Here, the results for α and b' are obtained from $\boldsymbol{f} = [0.11639, 0.25275, 0.15615, 0.29985, 0.17486]'$. We will consider naive portfolio (i.e. equally weighted) as the benchmark in the next section. The results under the naive portfolio as the benchmark are parallel to what we will see in this section. Finally, from the above parametrization, we obtain the sum of minimum variance strategies $Q = 1$ and $r - \alpha + b'\tilde{\zeta} = 0$.

As for the constraints, we set $\kappa = 1$ under the borrowing constraints. That is, we consider the case when borrowing is *prohibited* and $\kappa = Q$. We use the term borrowing constraints instead of borrowing prohibition when interpreting the results. On the other hand, under the rectangular constraints we set $l = [0, 0, 0, 0, 0]'$ and $u = [1, 1, 1, 1, 1]'$, implying that short selling is prohibited and the highest leverage for investment into a single stock is equal to 1. That is, an investor may use leverage at the portfolio level but not at the stock level. Moreover, we remind the reader that we use the growth optimal strategy, the probability maximizing strategy, and the reward maximizing strategy for the optimal investment strategies of the first, second and the third problems respectively.

We first provide the optimal fictitious parameter values in Table 1 below. We remind that $\nu^* = \mathbf{0}_N$ when the markets are unconstrained. We see that the optimal fictitious parameter under the borrowing constraints (which is ν_1^*) for three problems are the same. The equality is due to our parametrization by setting $\kappa = Q = 1$. For example, for a small change in κ by setting $\kappa = 1.1$, we obtain $\nu_1^* = -0.1552$ for the growth optimal portfolio, $\nu_1^* = -0.0515$ for the probability maximizing portfolio and $\nu_1^* = -0.1555$ for the reward maximizing portfolio[29]. Under the rectangular constraints, ν^* is different in each case.

Given the optimal fictitious parameter values, we then compute the growth rate for each portfolio strategy in Table 2. We observe positive values in all problems and investment cases considered in the study. For example, under the unconstrained markets, the growth rate of the growth optimal portfolio is $\theta = 1.3609 > 0$. The growth rates, θ_{ν^*}, pertaining to the same portfolio under the constrained markets are also positive. Therefore, the markets are favorable and the growth optimal strategy minimizes the expected time the ratio process, $Z^{\boldsymbol{w}}(\cdot)$, hits U. However, we may deduce from the first line of (4.6) that the aforementioned objective is

[28]The benchmark may be a traded portfolio since we consider $\beta \neq 0$ in the examples.

[29]Note that we have $r - \alpha + b'\tilde{\zeta} = 0$. Then, for the second problem, when we set $\kappa = 1.1 > Q = 1$ and $\beta = 0$, we obtain $\nu_1 = 0.4533 > 0$, implying no solution under the borrowing constraints since we must have $\nu_1 < 0$ when the constraints are binding. The result follows from our claim in Section 5.2.

Table 1: Optimal Fictitious Parameters (ν^*)

Portfolio Strategy	Borrowing Constraints (ν_1^*)	Rectangular Constraints (ν^*)
Growth Optimal Portfolio	-0.1628	$[0, 0.0552, 0.2073, 0.1128, 0.1083]'$
Probability Maximizing Portfolio	-0.1628	$[0, 0, 0.1133, 0.0822, 0.0070]'$
Reward Maximizing Portfolio	-0.1628	$[0, 0.0902, 0.2200, 0.1231, 0.1152]'$

fulfilled at a later time with stricter constraints. This is evident from the declining growth rate values in the first row of Table 2. Thus, as the constraints become stricter, the expected time to beat a benchmark becomes longer.

Table 2: Growth Rates of Optimal Portfolio Processes

Portfolio Strategy	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints
Growth Optimal Strategy	1.3609	1.1863	0.1700
Probability Maximizing Strategy	0.1846	0.1715	0.1000
Reward Maximizing Strategy	1.3591	1.1842	0.1656

There are also declines in the growth rates under the two other portfolio strategies. The decline with the probability maximizing strategy is interesting. On the one hand, constraints take effect, preventing the investor to take full advantage of the favorable markets, on the other hand, as we observe in Table 3, γ^- increases slowly toward -1 as the restrictions become stricter, implying that the strategy becomes closer to the growth optimal strategy; thus, the growth rate might increase. However, the growth rate still declines as it seems constraints have stronger effect on the values[30] of the example we consider here. Nonetheless, the positive growth rates are still an advantage to an investor despite the constraints. For example, we see in the third row of Table 2 that thanks to the positive growth rate, an investor will be able to maximize her expected discounted reward[31]. In any case, the growth rates are the lowest under the probability maximization strategy. Growth rates under the growth optimal and reward maximizing strategies are, on the other hand, very close. The similarity of results may be due to the values of $\hat{\eta}_b$.

We see from Table 3 that, under the unconstrained and borrowing constrained markets, $\hat{\eta}_b$ is just a bit larger than -1. Notice from the specifications in (6.3), for $\hat{\eta}_b = -1$, the strategy takes the form of the growth optimal strategy. Therefore, the similarity of the growth rates obtained by pursuing the growth optimal strategy and the reward maximizing strategy is not a surprise under the two cases. However, under the case of rectangular constraints, the value

[30]For example, in the third example, even if the results are not reported, we note here that the growth rate under the probability maximizing strategy increases as constraints take effect. When the constraints bind, γ^- increases toward -1 and the growth rate of the survival probability maximizing portfolio increases. Therefore, growth does not necessarily decline as constraints take effect, because the level of the growth rate is also dependent on the values of α and b'.

[31]Notice that for the third problem the markets are favorable under the borrowing constraints, since $\theta_{\nu^*} - \frac{1}{2}\left(1+\frac{1}{\hat{\eta}_b}\right)^2\left(\|\zeta\|^2 - \frac{D^2}{K}\right) = 1.1842 > 0$, implying that the markets are favorable in the sense mentioned in Theorem 6.1. Furthermore, from Table 3, we have $-1 < \hat{\eta}_b < 0$. Therefore, the problem solved here is maximizing the expected discounted reward. The same is true for the case of rectangular constraints.

Table 3: Optimal γ^- and $\hat{\eta}_b$

Portfolio Strategy	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints
Probability Maximizing Strategy (γ^-)	-14.7484	-13.8361	-11.6215
Reward Maximizing Strategy ($\hat{\eta}_b$)	-0.9646	-0.9596	-0.7391

of $\hat{\eta}_b$ is a bit more larger than -1, yet the growth rates of the two strategies are still very close.

Observing Table 2 again, we see that the differences between the growth rates of the probability maximizing strategy and the growth optimal strategy across all cases are considerably large. This is because the values of γ^- in Table 3 are much smaller than -1[32]. That is, with $\gamma^- \downarrow -\infty$, or $\hat{\eta}_b \uparrow 0$, the growth rate of the benchmarked portfolio becomes lower than that would be obtained under the growth optimal strategy. Especially, the diversion under the probability maximizing strategy is larger because the difference of γ^- from -1 is larger[33]. Moreover, we also see that the stricter the constraints are, the larger are the values of γ^- and $\hat{\eta}_b$ in Table 3. This in turn implies that the optimal values in (5.4) and (6.2) become smaller[34] as the constraints impose more limitations on the investment strategies.

We proceed to the analysis of the optimal investment strategies. We start with the analysis of the growth optimal strategy whose values under different scenarios are given in Table 4. We see that risk taking decreases significantly (from 3.1451 in unconstrained markets to 0.5654 under the rectangular constraints) with stricter constraints. However, while under the unconstrained and borrowing constrained markets, all stocks are invested in some form, under the rectangular constraints only the first stock is invested and the investment is in the form of buying. That stock has the highest capital allocation in all cases, because its market price of risk is the highest (see the final column of Table 4). As a second point, we observe under the unconstrained markets that the stock with the most negative market price of risk is short sold the most (observe that the value is equal to -10.1511). Therefore, when there are no constraints, an investor pursuing the growth optimal strategy invests the most (either in the form of buying or short-selling) in stocks that provide the highest growth potential.

Table 4: Growth Optimal Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	13.5248	13.0322	0.5654	1.0873
w_2^*	4.3411	4.0469	0	0.3959
w_3^*	-10.1511	-4.0759	0	-1.0300
w_4^*	-5.0275	-9.0087	0	-0.1218
w_5^*	0.4579	-2.9944	0	0.2399
Sum	3.1451	1.0000	0.5654	-

When borrowing is constrained, *investment exposure* (i.e. $|\boldsymbol{w}|$) to first two stocks declines

[32]We already mentioned that from the specification in (5.5) when $\gamma^- = -1$, probability maximizing strategy becomes equal to the growth optimal startegy.

[33]We remind that we have $\gamma^- < -1$ because, as mentioned in the sequel, the markets are favorable.

[34]Observe that the probability function in (5.4) decreases with increasing γ^- and the optimal expected discounted reward function in (6.2) decreases with increasing $\hat{\eta}_b$.

slightly. The decline is much higher in the third stock, the stock with the most negative market price of risk value. On the other hand, the investment exposure to fourth and fifth stocks increases. Especially, the form of the investment exposure in the fifth stock changes from buying to short-selling. That is, in this example, we see that when borrowing is constrained, exposure to the stocks with the highest *absolute value* of market price of risk (i.e. $|\tilde{\zeta}|$) declines. This decline is met with an increase in the investment exposure to the stocks with the lowest absolute value of market price of risk. Finally, under the rectangular constraints investment is done only in the first stock. The finding is interesting, because we could expect investment in the first, second and fifth stocks, while the investment in the third and fourth stocks to be stopped since short-selling for each stock is prohibited. However, the optimal result yields investment only in the first stock.

Table 5: Probability Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	1.0255	1.0499	0.6475	1.0873
w_2^*	0.5300	0.5270	0.3591	0.3959
w_3^*	-0.5427	-0.1497	0	-1.0300
w_4^*	-0.0614	-0.3729	0	-0.1218
w_5^*	0.1940	-0.0542	0	0.2399
Sum	1.1454	1.0000	1.0066	-

In Table 5, the values of the probability maximizing strategy are given. When the markets are unconstrained, the level of risk taken under the probability maximization is less than that is taken under the growth optimal strategy (compare 3.1451 in Table 4 with 1.1454 in Table 5). In effect, the exposure in each stock under the probability maximization is relatively much less than the exposure under the growth maximization. The same is true under the borrowing constraints as well, even if the sum of the strategies is the same. That is, probability maximizing strategy is more timid than the growth maximizing strategy under the unconstrained and borrowing constrained markets. This result may be due our choice of parameter values, because, we will see in the third example that the new choices for α and b' makes the same strategy bolder. In fact, before introducing that example, we may already see here that the probability maximizing strategy is not more timid under the rectangular constraints. In contrast, it is bolder as the investor invests both in the first and second stock with higher exposure than she does under the growth optimal strategy. These are the stocks with the highest market price of risk values and the sum of exposures to these stocks is just above 1.

In Table 6, we observe the optimal investment strategy values of the reward maximization problem. Different from the results of the previous problem, the investment exposure in each stock is higher than that of the growth optimal strategy under the unconstrained and borrowing constrained markets. Furthermore, we also observe that the risk level taken under the unconstrained markets with the strategy is also slightly higher than the risk level taken with the growth optimal strategy (compare total exposure value 3.1451 in Table 4 with the total exposure value 3.2239 in Table 6). This result as well may be due to our choice of parameter values, because in the third example for our new choices of α and b', the reward maximizing strategy is slightly more timid than the growth optimal strategy when $\hat{\eta}_b$ is more closer to -1. Finally, under the case of rectangular constraints, an investor invests only in the first stock as

Table 6: Reward Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	14.0174	13.5765	0.3905	1.0873
w_2^*	4.4913	4.2068	0	0.3959
w_3^*	-10.5298	-4.2543	0	-1.0300
w_4^*	-5.2233	-9.4011	0	-0.1218
w_5^*	0.4683	-3.1280	0	0.2399
Sum	3.2239	1.0000	0.3905	-

she does under the growth optimal strategy. However, the exposure is less than that of the growth optimal strategy. Therefore, the investment strategy in this case is timid.

7.2 The Naive Benchmark Portfolio Strategy

Here, we let other parameters be the same and set $\boldsymbol{f} = [0.2, 0.2, 0.2, 0.2, 0.2]^{'}$. From this, we obtain the equalities $\alpha = 0.11600$, $b = [0.22000, 0.32400, 0.31400, 0.23800, 0.25400]^{'}$ and we still have $Q = 1$. Since the growth optimal strategy is independent of the benchmark parameters, the optimal investment values for the first problem do not change. The only interesting cases correspond to the optimal strategies of the second and third problems. We start by reporting the optimal weights of the second problem in Table 7.

Table 7: Probability Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	1.1056	1.1300	0.6684	1.0873
w_2^*	0.4815	0.4788	0.2912	0.3959
w_3^*	-0.5035	-0.1099	0	-1.0300
w_4^*	-0.1553	-0.4674	0	-0.1218
w_5^*	0.2175	-0.0315	0.0578	0.2399
Sum	1.1458	1.0000	1.0174	-

We observe the similarity of the results in the above table with those in Table 5. In all cases the signs of the investment strategies are the same. In addition, the investment exposures under the unconstrained and borrowing constrained cases are more or less similar. Under the rectangular case, there is an extra level of exposure. We see in Table 7 that the total investment exposure increases slightly ($1,0174$ in Table 7 vs. $1,0066$ in Table 5) and the investor becomes exposed to the fifth stock with an investment weight of $5,78\%$ as opposed to $0,00\%$ in Table 5. Finally, we still observe under the unconstrained and borrowing constrained cases that the investment exposure is larger to stocks with higher absolute value of the market price of risk. Under the rectangular constraints, the investor invests only in stocks with positive market price of risk by allocating largest capital starting with the stock that has the highest market price of risk.

Table 8: Reward Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	14.0168	13.5762	0.3745	1.0873
w_2^*	4.4940	4.2100	0	0.3959
w_3^*	-10.5334	-4.2572	0	-1.0300
w_4^*	-5.2206	-9.3992	0	-0.1218
w_5^*	0.4674	-3.1298	0	0.2399
Sum	3.2243	1.0000	0.3745	-

In Table 8, we observe the values for the third problem. We see that the results for the reward maximizing weights are very close to those in Table 6. We don't comment further on the results, since they follow from the previous section. In sum, small changes in the proportional trading strategies of the benchmark portfolio process (from random to equal weights) do not seem to affect the results substantially. We will see in the next section that a large change in α and b' (i.e. increasing benchmark process' return and risk level) has a somewhat noticeable effect on the results.

7.3 Effect of the Increase in α and b'

In this example, we consider a benchmark different from a portfolio process traded with strategy $\boldsymbol{f}$. We let $\alpha = 23.000$, $b = [11.5400, 3.7778, -3.3247, 1.1000, 2.4000]'$ and take other exogenous parameters the same. With the new values, we obtain $Q = 35.8639$. Here as well the optimal investment weights of the first problem remain unchanged as the growth optimal strategy is independent of the benchmark parameters. The optimal weights for the second problem are given in Table 9 below.

Table 9: Probability Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	80.2503	46.0498	1.0000	1.0873
w_2^*	28.8767	15.3095	1.0000	0.3959
w_3^*	-83.1250	2.8250	0	-1.0300
w_4^*	-6.0819	-40.5191	1.0000	-0.1218
w_5^*	13.4037	-22.6652	0.8575	0.2399
Sum	33.3239	1.0000	3.8575	-

Comparing the results in Table 9 with those in Table 5 shows a substantial difference in risk taking under the unconstrained and rectangular constrained markets. The increase in risk taking is around 33 times under the unconstrained market, while it is around 3.8 times under the rectangular constraints. The investment under the rectangular constraints is done to the first, second and third stocks at the constraint level. Investment in the fifth stock is a bit lower than the full investment strategy level of 1. Only the third stock, the one with negative market price of risk value is not invested. Finally, the total exposure level under the unconstrained market is closer to Q (compare 33.3239 in Table 9 with $Q = 35.8639$) with the increase in α and b'. We will comment more on this after the analysis of the reward maximization problem's results given in Table 10 below.

Table 10: Reward Maximizing Strategy

Optimal Weights ($\boldsymbol{w}^*$)	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints	Market Price of Risk ($\tilde{\zeta}$)
w_1^*	13.4655	12.9789	0.4355	1.0873
w_2^*	4.3193	4.0287	0.0027	0.3959
w_3^*	-10.0863	-4.0870	0.0645	-1.0300
w_4^*	-5.0266	-8.9579	0.0005	-0.1218
w_5^*	0.4464	-2.9627	0.0298	0.2399
Sum	3.1183	1.0000	0.5330	-

We first notice in Table 10 that the optimal values under the unconstrained and borrowing constrained markets cases are quite similar to those of the growth optimal strategy in Table 4. Second, investment under the rectangular markets slightly increased as opposed to the investment level in the first example (see Table 6). In addition, all stocks are more or less invested this time. Therefore, with the increase in α and b' diversification took place under the rectangular constraints. The total exposure still remained below that of the growth optimal strategy. The similarity of the values under the unconstrained and borrowing constrained case may be due to the values of $\hat{\eta}_b$ we see below.

Table 11: Optimal γ^- and $\hat{\eta}_b$

Portfolio Strategy	Unconstrained Markets	Borrowing Constraints	Rectangular Constraints
Probability Maximizing Strategy (γ^-)	-12.8811	-2.0380	-1.8098
Reward Maximizing Strategy ($\hat{\eta}_b$)	-0.999181	-0.999179	-0.999174

The values of $\hat{\eta}_b$ in all cases are very close to -1, the value that equates the reward maximizing strategy with the growth optimal one. That may be one possible explanation to the similarity of the results. Furthermore, the investment exposures with the reward maximizing strategy under the unconstrained and borrowing constrained markets are generally slightly lower than those of the growth optimal strategy (see Table 4). Interestingly, we observe an instance when the reward maximizing strategy is more timid than the growth optimal strategy for $-1 < \hat{\eta}_b$ very close to -1 (favorable markets). Furthermore, we observe that the values of γ^- are relatively smaller than -1 in all cases. Especially, under the unconstrained market case, γ^- is much smaller, but the risk exposure is much higher than that under the growth optimal strategy. More clearly, even if $\gamma^- < -1$ (favorable markets), as we claimed in the interpretations of the results of the second problem, we see that an investor might still be able to pursue bolder strategies (compare 33.3239 in Table 9 with the growth optimal exposure of 3.1451 in Table 4.).

To see why, we remind the reader to compare the total exposure under the probability maximizing strategy in Table 9 with that of the minimum variance strategy (33.3239 in Table 9 vs. $Q = 35.8639$). We saw that $\gamma^- \downarrow -\infty$ for the probability maximizing strategy implies

$\boldsymbol{w} \rightarrow (\sigma^{-1})' b$. Since the exposure under the minimum variance strategy is quite large when compared with that of the growth optimal strategy, then the probability maximizing strategy becomes bolder as γ^{-} becomes smaller. In sum, while there may be instances when the probability maximizing strategy is more timid than the growth optimal strategy under the favorable markets (see the first example), there may also be instances (i.e. substantial change in α and b') when the probability maximizing strategy is bolder under the same market conditions. As a result, favorability of the markets does not necessarily ensure timid strategies when maximizing the probability of success or bolder strategies when maximizing expected discounted reward with respect to a benchmark.

8 Conclusion

In this paper, we solved three problems concerned with the performance measurement of a portfolio manager against a benchmark in constrained markets. The constraints we consider are namely the borrowing and rectangular constraints. The performance measures are conceptualized under three problems which are namely: (i) the minimization (maximization) of the time to beat (stay above) a benchmark; (ii) the maximization of the probability of beating a benchmark before incurring a shortfall; and (iii) the maximization (minimization) of the expected discounted reward (penalty) with respect to a benchmark. We solve these problems analytically under the borrowing constraints. As for the solutions under the rectangular constraints, we employ a computational procedure. In all cases, we establish the conditions for the existence of the solutions and conduct an analysis to see the effect of changes in key parameters on the optimal results. This includes devising the change in results when the markets are complete and discovering the effect of constraints on optimal values. Finally, we conduct a numerical analysis by considering three examples. These examples help us to see clearly the effect of constraints on the investment behaviour of an investor trading in favorable markets.

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Appendix

A The Main Argument for The Auxiliary Market

We will show that $Z_\nu^{\boldsymbol{w}}(t) \geq Z^{\boldsymbol{w}}(t)\ \forall \nu \in \tilde{\mathcal{K}}$ Lebesgue almost everywhere for $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$ by following Karatzas and Shreve (1998). We start by defining

$$\mathcal{Z}(t) = Z_\nu^{\boldsymbol{w}}(t) - Z^{\boldsymbol{w}}(t). \tag{A.1}$$

We can rewrite $\mathcal{Z}(t)$, for $t < \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}$, as

$$\begin{aligned}\mathcal{Z}(t) := &\int_0^t \mathcal{Z}(s)(\hat{r} + \delta(\nu))\mathrm{d}s + \int_0^t \mathcal{Z}(s)\boldsymbol{w}'(s)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\mathrm{d}s \\ &+ \int_0^t \mathcal{Z}(s)(\boldsymbol{w}'(s)\sigma - b')\mathrm{d}B^\bullet(s) - \int_0^t \mathcal{Z}(s)\,\beta\,\mathrm{d}B^{(N+1)}(s) \\ &+ \int_0^t Z^{\boldsymbol{w}}(s)\left(\delta(\nu) + \boldsymbol{w}'(s)\nu\right)\mathrm{d}s.\end{aligned} \tag{A.2}$$

Next, we define

$$\begin{aligned}\mathcal{H}(t) := \exp\Bigg\{ &\int_0^t -(\hat{r} + \delta(\nu))\mathrm{d}s - \int_0^t \boldsymbol{w}'(s)(\hat{\mu} + \nu(t) - r\underset{\sim}{1})\mathrm{d}s \\ &- \int_0^t (\boldsymbol{w}'(s)\sigma - b')\mathrm{d}B^\bullet(s) + \int_0^t \beta\,\mathrm{d}B^{(N+1)}(s) \\ &- \frac{1}{2}\int_0^t (\|\boldsymbol{w}'(t)\sigma - b'\|^2 + \beta^2)\mathrm{d}s\Bigg\}.\end{aligned} \tag{A.3}$$

Then, it follows that

$$\mathrm{d}\left(\mathcal{Z}(t)\mathcal{H}(t)\right) = \mathcal{H}(t)\tilde{Z}^{\boldsymbol{w}}(t)\left(\delta(\nu) + \boldsymbol{w}'(t)\nu\right)\mathrm{d}t. \tag{A.4}$$

The above equation is non-negative since $\mathcal{Z}(0) = 0$, $\mathcal{H}(t), Z^{\boldsymbol{w}}(t) \geq 0$, and by definition, $\delta(\nu) + \boldsymbol{w}'\nu \geq 0\ \forall \nu \in \tilde{\mathcal{K}}$ Lebesgue almost every $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$. This induces to $Z_\nu^{\boldsymbol{w}}(t) \geq Z^{\boldsymbol{w}}(t)$. Equality is attained when we find the vector of optimal fictitious parameters.

The inequality $\delta(\nu) + \boldsymbol{w}'\nu \geq 0\ \forall \nu \in \tilde{\mathcal{K}}$ implies that the set of admissible investment strategies under the constrained market, denoted by $\mathcal{A}_c$, is $\mathcal{A}_c \subseteq \mathcal{A}_\nu$ and hence $V_\nu(z) \geq V_{\nu^*}(z)$ for the maximization problem and $V_\nu(z) \leq V_{\nu^*}(z)$ for the minimization problem $\forall \nu \in \tilde{\mathcal{K}}$. Therefore, we have the equality when the optimal fictitious parameter $\nu^* \in \tilde{\mathcal{K}}$ and the optimal investment strategy $\boldsymbol{w}^* \in \mathcal{K}$ gives $\delta(\nu^*) + {\boldsymbol{w}^*}'\nu^* = 0$. It follows that $Z_{\nu^*}^{\boldsymbol{w}}(t) = Z^{\boldsymbol{w}}(t)$ Lebesgue almost every for $t \in [0, \tau_L^{\boldsymbol{w}} \wedge \tau_U^{\boldsymbol{w}}]$. Thus, for the maximization and minimization problems, we can obtain ν^* by respectively solving

$$\nu^* := \arg\inf_{\nu \in \mathcal{D}} V_\nu(z), \qquad \nu^* := \arg\sup_{\nu \in \mathcal{D}} V_\nu(z).$$

B Proof of Theorem 3.1

Here we provide the proof for the maximization problem. The proof of the minimization problem can also be done similarly. Let the HJB equation that G_ν satisfies for all $\nu \in \tilde{\mathcal{K}}$ be

$$
\begin{aligned}
-\rho(z)G_\nu + q(z) + \sup_{\boldsymbol{w}} \Big\{ & \Big((\hat{r} + \delta(\nu)) + \boldsymbol{w}'(\hat{\mu} + \nu - r\underset{\sim}{1})\Big) zG_\nu' \\
& + \frac{1}{2}\left(\boldsymbol{w}'\Sigma\boldsymbol{w} - 2\boldsymbol{w}'\sigma b + b'b + \beta^2\right) z^2 G_\nu''\Big\} = 0. \qquad \text{(B.1)}
\end{aligned}
$$

From the first-order condition, we obtain the maximizing control policy as

$$
\boldsymbol{w}^*(z) = (\sigma^{-1})'b - (\sigma^{-1})'\zeta_\nu(z)\frac{G_\nu'}{zG_\nu''}. \qquad \text{(B.2)}
$$

Then, we substitute the above into (B.1) and rearrange the terms to write the partial differential equation that G_ν satisfies

$$
-\rho(z)G_\nu + q(z) + (\hat{r} + \delta(\nu) + b'\zeta_\nu)zG_\nu' - \frac{1}{2}\|\zeta_\nu\|^2\frac{(G_\nu')^2}{G_\nu''} + \frac{1}{2}\beta^2 z^2 G_\nu'' = 0. \qquad \text{(B.3)}
$$

Next, we show that the above results are optimal. To this end, we introduce for $\nu \in \tilde{\mathcal{K}}$

$$
\begin{aligned}
M_\nu(t, Z_\nu^{\boldsymbol{w}}(t)) \quad = \quad & \exp\left\{-\int_0^t \rho(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s\right\} G_\nu(Z_\nu^{\boldsymbol{w}}(t)) \\
& + \int_0^t \exp\left\{-\int_0^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\} q(Z_\nu^{\boldsymbol{w}}(s))\mathrm{d}s. \qquad \text{(B.4)}
\end{aligned}
$$

Moreover, we fix $x \in (L, U)$ and define the stopping time

$$
\tau_n^{\boldsymbol{w}} := \tau^{\boldsymbol{w}} \wedge \inf\{u > t \mid |Z_\nu^{\boldsymbol{w}}(u) - Z_\nu^{\boldsymbol{w}}(t)| \geq n\} \quad \text{for} \quad n \in \mathbb{N}.
$$

Then, we choose an admissible strategy $\boldsymbol{w}(\cdot) \in \mathcal{A}_\nu(x)$ and write by the application of Itô's formula for $t \leq \tau_n^{\boldsymbol{w}}$

$$
\begin{aligned}
& M_\nu\left(\tau_n^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}})\right) = M_\nu\left(t, Z_\nu^{\boldsymbol{w}}(t)\right) \\
& + \int_t^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_t^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\}\{-\rho(Z_\nu^{\boldsymbol{w}})G_\nu\left(Z_\nu^{\boldsymbol{w}}(s)\right) + q(Z_\nu^{\boldsymbol{w}}(s)) + \mathcal{L}_\nu^{\boldsymbol{w}}G_\nu\left(Z_\nu^{\boldsymbol{w}}(s)\right)\}\mathrm{d}s \\
& \quad + \int_t^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_t^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\} Z_\nu^{\boldsymbol{w}}(s)\left(\boldsymbol{w}'(Z_\nu^{\boldsymbol{w}}(s))\sigma - b'\right) G_\nu'\left(Z_\nu^{\boldsymbol{w}}(s)\right)\mathrm{d}B^\bullet(s) \\
& \quad - \int_t^{\tau_n^{\boldsymbol{w}}} \exp\left\{-\int_t^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\} Z_\nu^{\boldsymbol{w}}(s)\beta G_\nu'\left(Z_\nu^{\boldsymbol{w}}(s)\right)\mathrm{d}B^{N+1}(s). \qquad \text{(B.5)}
\end{aligned}
$$

After taking the conditional expectation of (B.5), we may see that the stochastic integrals vanish because we have $e^{-\int_t^{\tau_n^{\boldsymbol{w}}} \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u} \in (0, 1]$, continuous G_ν, admissible $\boldsymbol{w}(\cdot)$, and bounded $Z_\nu^{\boldsymbol{w}}$ on $[t, \tau_n^{\boldsymbol{w}}]$. That is,

$$
\begin{aligned}
\mathbb{E}_z\left[\int_t^{\tau_n^{\boldsymbol{w}}}\left\|\exp\left\{-\int_t^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\} Z_\nu^{\boldsymbol{w}}(s)\left(\boldsymbol{w}'(Z_\nu^{\boldsymbol{w}}(s))\sigma - b'\right) G_\nu'\left(Z_\nu^{\boldsymbol{w}}(s)\right)\right\|^2\mathrm{d}s\right] \quad &< \quad \infty; \\
\mathbb{E}_z\left[\int_t^{\tau_n^{\boldsymbol{w}}}\left\|\exp\left\{-\int_t^s \rho(Z_\nu^{\boldsymbol{w}}(u))\mathrm{d}u\right\} Z_\nu^{\boldsymbol{w}}(s)\beta G_\nu'\left(Z_\nu^{\boldsymbol{w}}(s)\right)\right\|^2\mathrm{d}s\right] \quad &< \quad \infty.
\end{aligned}
\qquad \text{(B.6)}
$$

In addition, we write $\forall \boldsymbol{w} \in \mathbb{R}^N$, the inequality

$$-\rho(Z_\nu^{\boldsymbol{w}}(s))G_\nu\left(Z_\nu^{\boldsymbol{w}}(s)\right) + q(Z_\nu^{\boldsymbol{w}}(s)) + \mathcal{L}_\nu^{\boldsymbol{w}} G_\nu\left(Z_\nu^{\boldsymbol{w}}(s)\right) \leq 0 \tag{B.7}$$

$\mathbb{P}$-almost surely. We set $t = 0$ and from (B.4) and (B.5)-(B.7), we write

$$G_\nu(x) \geq \mathbb{E}_z\left[M_\nu\left(\tau_n^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}})\right)\right]. \tag{B.8}$$

We have $\tau_n^{\boldsymbol{w}} \to \tau^{\boldsymbol{w}}$ as $n \to \infty$. Moreover, since $|G_\nu(x)| \leq c_1 + c_2 \ln\left(\frac{z}{L}\right) + c_3 \ln\left(\frac{U}{z}\right)$, we write from the dominated convergence theorem as $n \to \infty$

$$\mathbb{E}_z\left[M_\nu\left(\tau_n^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau_n^{\boldsymbol{w}})\right)\right] \to \mathbb{E}_z\left[M_\nu(\tau^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}}))\right].$$

It follows from the inequality in (B.8) that

$$G_\nu(z) \geq \mathbb{E}_z\left[M_\nu\left(\tau^{\boldsymbol{w}}, Z_\nu^{\boldsymbol{w}}(\tau^{\boldsymbol{w}})\right)\right] = J_\nu^{\boldsymbol{w}}(z). \tag{B.9}$$

By taking the supremum over all admissible strategies in the auxiliary market, we obtain $G_\nu(z) = V_\nu(z)$. Then, for $t < \tau^{\boldsymbol{w}^*}$, $\boldsymbol{w}^*(Z_\nu^{\boldsymbol{w}^*}(t))$ is the optimal investment strategy and $Z_\nu^{\boldsymbol{w}^*}(t)$ is the optimal benchmarked wealth process under the auxiliary market.

C Proof of Theorem 4.1

We provide the proof for the maximization problem. We will see at the end of this proof that the minimization problem can be solved similarly with a slight transformation of the objective function. For the proof, we take $\rho(\cdot) = 0$, $q(\cdot) = 1$, $H(\cdot) = 0$ in Equation 3.3. As a result, from Equation 3.3 of Theorem 3.1, the partial differential equation that $\underline{E}_\nu$ is a solution to is given by

$$1 + (\hat{r} + \delta(\nu) + b'\zeta_\nu)z\underline{E}'_\nu - \frac{1}{2}\|\zeta_\nu\|^2 \frac{(\underline{E}'_\nu)^2}{\underline{E}''_\nu} + \frac{1}{2}\beta^2 z^2 \underline{E}''_\nu = 0. \tag{C.1}$$

We first consider the proof under the borrowing constraints as the proof under the rectangular constraints follows from that of the borrowing constraints. Since under the borrowing constraints $\delta(\nu) = \sup_{\boldsymbol{w}}(-\boldsymbol{w}'\nu) = -\kappa\nu_1$, $\nu = \nu_1\underset{\sim}{1}$ and $\zeta_\nu = \zeta + \sigma^{-1}\nu_1\underset{\sim}{1}$ for a scalar $\nu_1 \leq 0$, optimizing (C.1) over $\nu \in \tilde{\mathfrak{K}}$ yields the minimizing fictitious parameter[35]

$$\nu_1^*(z) = \frac{1}{K}\left((-\kappa + Q)\frac{z\underline{E}''_\nu}{\underline{E}'_\nu} - D\right). \tag{C.2}$$

Substituting the above into (C.1) yields the partial differential equation

$$1 + \left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa + Q)\right)z\underline{E}'_{\nu^*} + \frac{1}{2}\left(\frac{1}{K}(-\kappa + Q)^2 + \beta^2\right)z^2\underline{E}''_{\nu^*}$$
$$- \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(\underline{E}'_{\nu^*})^2}{\underline{E}''_{\nu^*}} = 0, \tag{C.3}$$

[35]Note that this is the general form of the optimal fictitious parameter under the borrowing constraints. We will refer to this specification in the proofs of the next two problems.

subject to the boundary condition $\underline{E}_{\nu^*}(L) = 0$. Notice that the second value function in (4.6), which is $C^2((0,L))$ and of the form $\underline{E}'_{\nu^*} > 0$ with $\underline{E}''_{\nu^*} < 0$[36], satisfies the equality in (C.3). Then, by substituting the value function in the second line of (4.6) to (C.2), we obtain the vector of minimizing fictitious parameters as in the first line of (4.4). From Equation (3.4), the value in the first line of (4.4) and the value function in the second line of (4.6), the maximizing $\boldsymbol{w}^*$ is given as in the first line of (4.7) (which is the growth optimal strategy under the borrowing constraints). Then substituting the optimal fictitious parameters in the first line of (4.4) to (4.3), we obtain $\theta_{\nu^*} \in \mathbb{R} \setminus \{0\}$ as in the first line of (4.5). It follows that the benchmarked portfolio process is given by, for $0 \le t \le \tau_L^{\boldsymbol{w}^*}$,

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = Z(0)\exp\left\{\theta_{\nu^*}t + \tilde{\zeta}'_{\nu^*}B^{\bullet}(t) - \beta B^{(N+1)}(t)\right\}, \text{ where } \tilde{\zeta}_{\nu^*} = \tilde{\zeta} + \sigma^{-1}\left(\frac{\kappa - \tilde{D}}{K}\right)\underset{\sim}{1}. \quad \text{(C.4)}$$

From (C.4), we write $(1/t)\mathbb{E}\left[\log(Z_{\nu^*}^{\boldsymbol{w}^*}(t))\right] = \theta_{\nu^*}$. Note that when $\theta_{\nu^*} < 0$, $Z_{\nu^*}^{\boldsymbol{w}^*}(t) \to 0$ as $t \to \infty$. Then, for $\theta_{\nu^*} < 0$, $\tau_L^{\boldsymbol{w}^*} < \infty$ $\mathbb{P}$-almost surely. Since $\boldsymbol{w}^*$ is a constant vector, $\int_0^{\tau^{\boldsymbol{w}^*}} \|\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(s))\|^2\,\mathrm{d}s < \infty$ $\mathbb{P}$-almost surely, implying that $\boldsymbol{w} \in \mathcal{A}_{\nu^*}$. We may also observe that $\boldsymbol{w}^*\underset{\sim}{1} = \kappa$, implying $\boldsymbol{w}^* \in \mathcal{K}$. Furthermore, $\underline{E}_{\nu^*}$ is dominated by $c_1 + \frac{1}{\theta_{\nu^*}}\log(\frac{z}{L})$ for constant $c_1 > 0$ (see Theorem 3.1). Then, the rest of the proof follows from the verification argument in the proof of Theorem 3.1. As a result, $\underline{E}_{\nu^*}(z)$ is the optimal value function under the borrowing constraints. Furthermore, for $t < \tau_L^{\boldsymbol{w}^*}$, $\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(t))$ is the optimal investment strategy and $Z_{\nu^*}^{\boldsymbol{w}^*}(t)$ is the optimal benchmarked wealth process.

The vector of optimal fictitious parameters under the rectangular constraints is found by minimizing θ_ν in (4.3) over all $\nu \in \tilde{\mathcal{K}}$ since the value function of interest $\underline{E}_\nu(z)$ is given by $1/(|\theta_\nu|)\log(z/L)$ when the constraints are not binding. This is a maximization problem. Then, we need to minimize this function overall $\nu \in \mathcal{D}$ to find its constrained value. It follows that the vector of optimal fictitious parameters under the rectangular constraints can be found by minimizing θ_ν (since $\theta_\nu < 0$ under the maximization problem) overall $\nu \in \tilde{\mathcal{K}}$. That is, we have

$$\epsilon_1^* := \arg\min_{\nu \in \tilde{\mathcal{K}}}\left[r + u'\nu^- - l'\nu^+ - \alpha + \frac{1}{2}\left(\|\tilde{\zeta}_\nu\|^2 + \|b\|^2 + \beta^2\right)\right]. \quad \text{(C.5)}$$

Note that $\delta(\nu)$ in (4.3) is replaced by $u'\nu^- - l'\nu^+$[37] since we consider rectangular constraints. The solution to the problem in (C.5) is found computationally and we can proceed to the proof under the rectangular constraints as we did under the borrowing constraints. This entails replacing the vector of fictitious parameters with ϵ_1^* appropriately and computing $\boldsymbol{w}^*$ and θ_{ν^*} accordingly. The specifications of the optimal value function and the benchmarked portfolio then follow.

Finally, notice that the minimization problem, $\bar{E}_\nu(z) = \inf_{\boldsymbol{w}\in\mathcal{A}_\nu}\mathbb{E}_z[\tau_U^{\boldsymbol{w}}]$ may be written as $\bar{E}_\nu(z) = -\sup_{\boldsymbol{w}\in\mathcal{A}_\nu}\mathbb{E}_z\left[-\tau_U^{\boldsymbol{w}}\right]$. Then, we can apply what we did above to the function $\hat{F}(x) = -\bar{F}(x)$ for the proof of the minimization problem.

D Proof of Theorem 5.1

[36] Obviously, the equation satisfies the boundary condition $\underline{E}_{\nu^*}(L) = 0$.

[37] We let $l = (l_1, \ldots, l_N)'$ and $u = (u_1, \ldots, u_N)'$.

For this problem as well, we first provide the proof under the borrowing constraints. From Remark 3.1, the value function for the problem under consideration can be recovered by taking $\rho(\cdot) = q(\cdot) = 0, H(L) = 0$ and $H(U) = 1$ in Equation 3.1. As in the first problem, by using Equation (3.3) in Theorem 3.1, we can write the partial differential equation that P_ν satisfies, then optimize over $\nu \in \tilde{\mathcal{K}}$ to find the vector of optimal fictitious parameters. Finally, we substitute ν^* to obtain the non-linear partial differential equation that P_{ν^*} satisfies[38]:

$$\left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa + Q)\right) z P'_{\nu^*} + \frac{1}{2}\left(\frac{1}{K}(-\kappa + Q)^2 + \beta^2\right) z^2 P''_{\nu^*} \\ -\frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right) \frac{(P'_{\nu^*})^2}{P''_{\nu^*}} = 0. \tag{D.1}$$

To find the maximizing strategy and the minimizing fictitious parameters, we guess a solution for P_{ν^*} of the form $A_1 - A_2 z^{1+\gamma}$, where A_1 and A_2 are constants. Solving by using the boundary conditions $P_{\nu^*}(L) = H(L) = 0$ and $P_{\nu^*}(H) = H(U) = 1$, we get

$$A_1 - A_2 z^{1+\gamma} = \frac{L^{1+\gamma} - z^{1+\gamma}}{L^{1+\gamma} - U^{1+\gamma}}. \tag{D.2}$$

Substituting the derivatives of the above function into (D.1) and solving for γ yields

$$\gamma^{\pm} = \frac{-\left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa + Q)\right) \pm \sqrt{\Xi}}{\left(\frac{1}{K}(-\kappa + Q)^2 + \beta^2\right)}, \tag{D.3}$$

where Ξ is given in the first line of (5.3). For $\|\zeta\|^2 - \frac{D^2}{K} \geq 0$, (D.3) has two roots: $\gamma^- < 0 < \gamma^+$. We see that $P_{\nu^*}(z)$ is concave increasing in z only for $\gamma < 0$. Therefore, the candidate value function is as given in (5.4). Given the candidate value function, we can find the minimizing fictitious parameters by replacing the first and second order derivatives of (5.4) into Equation (C.2) of the first proof. Then, the vector of minimizing fictitious parameters under the borrowing constraints is given as in the first line of (5.1). Likewise, by replacing the value in the first line of (5.1) and first and second order derivatives of (5.4) into (3.4), we find the maximizer as specified in the first line of (5.5).

Next, substituting the maximizer as specified in the first line of (5.5) and ν^* specified in the first line of (5.1) into (2.9) yields after the application of Itô's formula

$$Z^{\boldsymbol{w}^*}_{\nu^*}(t) = Z(0) \exp\left[\left(\theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\gamma^-}\right)^2 \left(\|\zeta\|^2 - \frac{D^2}{K}\right)\right)t \\ - \frac{1}{\gamma^-}\zeta'_{\nu^*} B^{\bullet}(t) - \beta B^{(N+1)}(t)\right], \tag{D.4}$$

where θ_{ν^*} is given as in the first line of[39] (4.5) and $\zeta_{\nu^*} = \zeta + (\sigma^{-1})'\left(\frac{\gamma^-}{K}(-\kappa + Q) - \frac{D}{K}\right)\underset{\sim}{1}$. Next,

[38]We will omit the first steps for the sake of brevity in the proof of the second and the third problems as they are similar to what we did in the first problem. Especially, the specification of ν^* under the borrowing constraints (see Equation (C.2)) is the same across three problems.

[39] Note that for the rectangular constraints, the problem has similar solution with θ_{ν^*} replaced by the second line of (4.5). Then the deterministic trend of $Z^{\boldsymbol{w}^*}_{\nu^*}(t)$ becomes $\theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\gamma^-(\nu^*)}\right)^2 \|\zeta_{\nu^*}\|^2$ and ν^* is the optimal fictitious parameters under the rectangular constraints. We will see how we find the optimal fictitious parameter values under the rectangular constraints in the rest of the proof.

we define

$$\Theta(\gamma^-) := \theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\gamma^-}\right)^2 \left(\|\zeta\|^2 - \frac{D^2}{K}\right).$$

It is clear that when $\Theta(\gamma^-) > 0$, as $t \to \infty$, $\mathbb{P}_z(\tau_L^{\boldsymbol{w}^*} < \infty) \to 0$ and $\mathbb{P}_z(\tau_U^{\boldsymbol{w}^*} < \infty) \to 1$, implying that $\tau_U^{\boldsymbol{w}^*} < \infty$ $\mathbb{P}$-almost surely. The inequalities of the probabilities are reversed when $\Theta(\gamma^-) < 0$, implying that $\tau_L^{\boldsymbol{w}^*} < \infty$ $\mathbb{P}$-almost surely. Since $\boldsymbol{w}^*$ is a constant vector, we have $\int_0^{\tau^{\boldsymbol{w}^*}} \|\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(s))\|^2 \, \mathrm{d}s < \infty$ $\mathbb{P}$-almost surely. As a result, $\boldsymbol{w}^*(\cdot) \in \mathcal{A}_{\nu^*}$. We may also observe that $\boldsymbol{w}^* \underset{\sim}{1} = \kappa$, implying $\boldsymbol{w}^* \in \mathcal{K}$.

Finally, notice that since $|P_{\nu^*}(z)| < 1$, the verification follows from the verification argument provided in the proof of Theorem 3.1. Therefore, $P_{\nu^*}(z)$ is the optimal value function under the borrowing constraints. Furthermore, for $t < \tau^{\boldsymbol{w}^*}$, $\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(t))$ is the optimal investment strategy and $Z_{\nu^*}^{\boldsymbol{w}^*}(t)$ is the optimal benchmarked wealth process.

The procedure of the proof for the case under the rectangular constraints necessitates the use of the partial differential equation provided in Theorem 3.1. From (3.3), we recall the partial differential equation that P_ν satisfies:

$$(\hat{r} + \delta(\nu) + b'\zeta_\nu) z P_\nu' - \frac{1}{2}\|\zeta_\nu\|^2 \frac{(P_\nu')^2}{P_\nu''} + \frac{1}{2}\beta^2 z^2 P_\nu'' = 0. \tag{D.5}$$

Then, substituting the specification of $P_\nu(z)$ provided in (D.2), and rearranging the terms yield

$$\gamma^2\beta^2 + \gamma 2(\hat{r} + \delta(\nu) + b'\zeta_\nu) - \|\zeta_\nu\|^2 = 0. \tag{D.6}$$

The roots of the above function are given by

$$\gamma^{\pm}(\nu) = \frac{-\left(\hat{r} + \delta(\nu) + b'\zeta_\nu\right) \pm \sqrt{\left(\hat{r} + \delta(\nu) + b'\zeta_\nu\right)^2 + \beta^2\|\zeta_\nu\|^2}}{\beta^2}. \tag{D.7}$$

As in the case of the borrowing constraints, we have $\gamma^- < 0 < \gamma^+$. The appropriate root is therefore γ^-, because $P_\nu(z)$ is concave increasing in z only for γ^-.

Next, observe that $\partial P_\nu / \partial \gamma^- < 0$. That is, as $\gamma^- \to 0$, P_ν declines. Since, we need to minimize $P_\nu(\cdot)$ over $\nu \in \tilde{\mathcal{K}}$ to find ν^*, then the minimization of $P_\nu(\cdot)$ over $\nu \in \tilde{\mathcal{K}}$ is equivalent to the maximization of γ^- over $\nu \in \tilde{\mathcal{K}}$ as P_ν is declining in increasing γ^-. Therefore, $\inf_{\nu \in \mathcal{D}} P_\nu(z) \equiv \sup_{\nu \in \mathcal{D}} \gamma^-(\nu)$, and given the specification of $\gamma^-(\nu)$, we find the vector of fictitious parameters when the constraints bind by solving

$$\epsilon_2^* := \arg\sup_{\nu \in \tilde{\mathcal{K}}} \left[-\left(\hat{r} + u'\nu^- - l'\nu^+ + b'\zeta_\nu\right) - \sqrt{\left(\hat{r} + u'\nu^- - l'\nu^+ + b'\zeta_\nu\right)^2 + \beta^2\|\zeta_\nu\|^2} \right]. \tag{D.8}$$

The solution to the problem in (D.8) is found computationally and we can continue the proof under the rectangular constraints as we did in the borrowing constraints. This entails replacing the vector of fictitious parameters found under the borrowing constraints with ϵ_2^* appropriately and computing the optimal values accordingly (see for example Footnote 39).

E Proofs of Theorems 6.1 & 6.2

Here as well, we first prove the problem under the borrowing constraints. For the proof, we set (see Remark 3.1 again) $\rho(\cdot) = \rho$, $q(\cdot) = 0$ and $H(U) = 1$ (or $H(L) = 1$ for minimization) in (3.1). With the use of Equation (3.3) in Theorem 3.1, we may first write the partial differential equation that R_ν satisfies, then optimize over $\nu \in \tilde{\mathcal{K}}$ to find the vector of optimal fictitious parameters. Substituting ν^* (whose specification is already given in (C.2)) to (3.3) yields the non-linear partial differential equation that R_{ν^*} satisfies:

$$-\rho R_{\nu^*} + \left(\hat{r} + b'\zeta - \frac{D}{K}(-\kappa + Q)\right) z R'_{\nu^*} + \frac{1}{2}\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right) z^2 R''_{\nu^*} - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)\frac{(R'_{\nu^*})^2}{R''_{\nu^*}} = 0. \tag{E.1}$$

To proceed one step ahead we guess a solution for both $\bar{R}_{\nu^*}$ and $\underline{R}_{\nu^*}$ in the form $Cz^{\eta+1}$ where C and η are constants. Substitution of this solution to (E.1) yields the following cubic equation:

$$\hat{\eta}^3 + \Psi_1 \hat{\eta}^2 + \Psi_2 \hat{\eta} + \Psi_3 = 0, \tag{E.2}$$

where

$$\Psi_1 = \frac{\frac{1}{2}\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right) + \hat{r} + b'\zeta - \frac{D}{K}(-\kappa+Q)}{\frac{1}{2}\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right)};$$

$$\Psi_2 = \frac{\hat{r} + b'\zeta - \frac{D}{K}(-\kappa+Q) - \rho - \frac{1}{2}\left(\|\zeta\|^2 - \frac{D^2}{K}\right)}{\frac{1}{2}\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right)};$$

$$\Psi_3 = -\frac{\left(\|\zeta\|^2 - \frac{D^2}{K}\right)}{\left(\frac{1}{K}(-\kappa+Q)^2 + \beta^2\right)}.$$

We solve the cubic equation by modifying the method shown in Weisstein (2003) to our case. To this end, we consider the discriminants

$$\Delta_1 = \frac{3\Psi_2 - \Psi_1^2}{9}; \quad \Delta_2 = \frac{9\Psi_1\Psi_2 - 27\Psi_3 - 2\Psi_1^3}{54}.$$

We have three unequal real roots if and only if $\Delta_2^2 + \Delta_1^3 < 0$. It follows that if $\Delta_2 < 0$, the roots $\hat{\eta}_i$ for $i = 0, 1, 2$ are

$$\hat{\eta}_i = 2\sqrt{-\Delta_1}\cos\left(\frac{\phi + 2i\pi}{3}\right) - \frac{\Psi_1}{3}, \text{ where } \phi = \arccos\left(\frac{\Delta_2}{\sqrt{-\Delta_1^3}}\right). \tag{E.3}$$

Then, from the above, say that we have three roots of the form $\hat{\eta}_a < -1 < \hat{\eta}_b < 0 < \hat{\eta}_c$[40]. We will have R_{ν^*} concave increasing for $\hat{\eta}_b$ and convex decreasing for $\hat{\eta}_a$. From the boundary conditions, we may specify the value functions in the form

$$\underline{R}_{\nu^*}(z) = \left(\frac{L}{z}\right)^{\hat{\eta}_a + 1} \quad \text{and} \quad \bar{R}_{\nu^*}(z) = \left(\frac{z}{U}\right)^{\hat{\eta}_b + 1}, \tag{E.4}$$

[40] We change the subscript of the roots to $i = a, b, c$ from numeric values $i = 1, 2, 3$ because the ordering of the roots with respect to numeric values is not clear, i.e. we may have $\hat{\eta}_0 < -1 < \hat{\eta}_1 < 0 < \hat{\eta}_2$ as well as $\hat{\eta}_1 < -1 < \hat{\eta}_0 < 0 < \hat{\eta}_2$ depending on the value of the portfolio model parameters.

and the vector of minimizing fictitious parameters is given from (C.2) as in the first line of (6.1) in Theorem 6.1 for the maximization problem. For the minimization problem, the vector of maximizing fictitious parameters is given as in the first line of (6.4) in Theorem 6.2. Likewise by substituting the first and second order derivatives of (E.4) and the pertaining fictitious parameters in (6.1) and (6.4) into (3.4) we find the maximizing control policy $\{\boldsymbol{w}^*(t), t \geq 0\}$ as specified in the first line of (6.3) and the minimizing control policy $\{\boldsymbol{w}^*(t), t \geq 0\}$ as specified in the first line of (6.6).

Given the problem we are solving, by substituting the appropriate maximizing and minimizing control policies and fictitious parameters into (2.9), we obtain the specification[41] by the application of Itô's formula

$$Z_{\nu^*}^{\boldsymbol{w}^*}(t) = Z(0) \exp\Bigg\{ \left(\theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\hat{\eta}}\right)^2 \left(\|\zeta\|^2 - \frac{D^2}{K} \right) \right) t - \frac{1}{\hat{\eta}} \zeta_{\nu^*} B^{\bullet}(t) - \beta B^{N+1}(t) \Bigg\}, \tag{E.5}$$

where θ_{ν^*} is specified in the first line of[42] (4.5) and $\zeta_{\nu^*} = \zeta + (\sigma^{-1})'\left(\frac{\hat{\eta}}{K}(-\kappa + Q) - \frac{D}{K}\right)\underset{\sim}{1}$. Then, we define

$$\Theta(\hat{\eta}) := \theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\hat{\eta}}\right)^2 \left(\|\zeta\|^2 - \frac{D^2}{K} \right).$$

For $\Theta(\hat{\eta}) > 0$, we have, as $t \to \infty$, $\mathbb{P}_z(\tau_L^{\boldsymbol{w}^*} < \infty) \to 0$ and $\mathbb{P}_z(\tau_U^{\boldsymbol{w}^*} < \infty) \to 1$, implying that $\tau_U^{\boldsymbol{w}^*} < \infty$ $\mathbb{P}$-almost surely. Then, when $\Theta(\hat{\eta}) < 0$, we obtain $\tau_L^{\boldsymbol{w}^*} < \infty$ $\mathbb{P}$-almost surely. We see that $\boldsymbol{w}^*$ is a constant vector. It follows that $\int_0^{\tau^{\boldsymbol{w}^*}} \|\boldsymbol{w}^*(Z_{\nu^*}^{\boldsymbol{w}^*}(s))\|^2 \, \mathrm{d}s < \infty$ $\mathbb{P}$-almost surely. Thus, $\boldsymbol{w}^*(\cdot) \in \mathcal{A}_{\nu^*}$. We may also observe that $\boldsymbol{w}^*\underset{\sim}{1} = \kappa$, implying $\boldsymbol{w}^* \in \mathcal{K}$.

Finally, we recall from Theorem 3.1 that $R_{\nu^*}(z)$ is bounded above with $c_1 + c_2 \log\left(\frac{z}{L}\right) + c_3 \log\left(\frac{U}{z}\right)$, since $|R_{\nu^*}(z)| \leq 1$. Hence, we set $c_1 = 1$ and $c_2 = c_3 = 0$. The rest for the verification of optimality follows from the verification argument provided in the proof of Theorem 3.1.

For the proof under the rectangular constraints, we recall the partial differential equation provided in Theorem 3.1. From (3.3), the partial differential equation that R_ν satisfies is

$$-\rho R_\nu + (\hat{r} + \delta(\nu) + b'\zeta_\nu) z R_\nu' - \frac{1}{2}\|\zeta_\nu\|^2 \frac{(R_\nu')^2}{R_\nu''} + \frac{1}{2}\beta^2 z^2 R_\nu'' = 0. \tag{E.6}$$

Substituting the specification of $R_\nu(z)$ in the form $Cz^{\eta+1}$ above and rearranging the terms yield

$$\eta^3 \frac{\beta^2}{2} + \eta^2(\hat{r} + \delta(\nu) + b'\zeta_\nu + \frac{1}{2}\beta^2) + \eta(\hat{r} + \delta(\nu) + b'\zeta_\nu - \rho - \frac{1}{2}\|\zeta_\nu\|^2) - \frac{1}{2}\|\zeta_\nu\|^2 = 0. \tag{E.7}$$

We may solve the equation as we did under the borrowing constraints. We omit the steps to keep the exposition as brief as possible. The solution will yield again three roots $\hat{\eta}_i(\nu)$ for $i = 0, 1, 2$.

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[41] Note that we use $\hat{\eta}_a$ for the minimization problem and $\hat{\eta}_b$ for the maximization problem. Instead of providing a specification of $Z_{\nu^*}^*(\cdot)$ for each one of them, we provide a general form for the ratio process by using $\hat{\eta}$. We do the same for other specifications when necessary in the rest of the proof.

[42] From Footnote 39 in the previous proof, for the rectangular constraints, the problem has similar solution with θ_{ν^*} replaced by the second line of (4.5). Then, the deterministic trend of $Z_{\nu^*}^{\boldsymbol{w}^*}(t)$ becomes $\theta_{\nu^*} - \frac{1}{2}\left(1 + \frac{1}{\hat{\eta}(\nu^*)}\right)^2 \|\zeta_{\nu^*}\|^2$ and ν^* is the optimal fictitious parameters under the rectangular constraints. We will see how we find the optimal fictitious parameter values in the rest of the proof.

We optimize each root over $\nu \in \tilde{\mathcal{K}}$ and say we have $\hat{\eta}_a(\epsilon_3^{a*}) < -1 < \hat{\eta}_b(\epsilon_3^{b*}) < 0 < \hat{\eta}_c(\nu^*)$, where $\epsilon_3^{a*} := \arg\sup_{\nu\in\tilde{\mathcal{K}}} \hat{\eta}_a(\nu)$ and $\epsilon_3^{b*} := \arg\inf_{\nu\in\tilde{\mathcal{K}}} \hat{\eta}_b(\nu)$. From the proof of the case with the borrowing constraints, we see that $\hat{\eta}_a(\epsilon_3^{a*})$ is for the minimization problem and $\hat{\eta}_b(\epsilon_3^{b*})$ is for the maximization problem, since R_{ν^*} is concave increasing for $\hat{\eta}_b(\nu^*)$ and convex decreasing for $\hat{\eta}_a(\nu^*)$. In that respect, we have the equivalence condition: $\sup_{\nu\in\mathcal{D}} \underline{R}_\nu(z) \equiv \sup_{\nu\in\mathcal{D}} \hat{\eta}_a(\nu)$ and $\inf_{\nu\in\mathcal{D}} \bar{R}_\nu(z) \equiv \inf_{\nu\in\mathcal{D}} \hat{\eta}_b(\nu)$. Then, the proof under the rectangular constraints follows similarly; we may replace the vector of fictitious parameters found under the borrowing constraints with ϵ_3^{i*} for $i = a, b$ appropriately and compute the optimal values.