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WAVELET VARIANCE RATIO TEST AND WAVESTRAPPING FOR THE DETERMINATION OF THE COINTEGRATION RANK

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Abstract

In this paper, I propose a wavelet based cointegration test for fractionally integrated time series. This proposed test is non-parametric and asymptotically invariant to different forms of short run dynamics. The use of wavelets allows one to take advantage of the wavelet based bootstrapping method particularly known as *wavestrapping*. In this regard, I introduce a new wavestrapping algorithm for multivariate time series processes, specifically for cointegration tests. The Monte Carlo simulations indicate that this new wavestrapping procedure can alleviate the severe size distortions which are generally observed in cointegration tests with time series containing innovations that possess highly negative MA parameters. Additionally, I apply the proposed methodology to analyse the long run co-movements in the credit default swap market of European Union countries.

Keywords: Fractional integration, Cointegration, Wavelet, Wavestrapping

1 Introduction

After the seminal work of Granger (1981), researchers were extensively concerned with the cointegration tests to find the number of long run relations among integrated variables. Barring a few exceptions, most of these tests are devised in the time domain. However, the frequency domain properties of integrated variables also have consequential implications in statistical analysis. For instance, Granger (1966) had observed that nonstationary variables have typical power spectra in which the low frequency components have dominating importance. Following the Granger's (1966) observation, a few scholars used the Fourier transform to construct spectral unit root tests (e.g. Choi and Phillips (1993), Robinson (1994) and Chambers et al. (2014)) and frequency domain cointegration tests (e.g. Levy (2002), Morana (2004) and Nielsen (2004)). Nonetheless, Fan and Gencay (2010) claim that the Fourier transform is not very appropriate when dealing with nonstationary variables since it lacks time resolution and only provides frequency domain information. Moreover, the authors point out that most of time series in Economics and Finance have complicated behaviors which cannot be analyzed by the Fourier transformation (Fan and Gencay, 2010). To resolve this issue, Fan and Gencay (2010) and Trokić (2016) exploit the wavelet filters and develop new spectral methods for testing unit roots. Similarly, I employ these spectral techniques in a multivariate setting and propose a new wavelet based nonparametric method for testing cointegration in fractionally integrated systems.

The method I proposed combines the Nielsen's (2010) variance ratio cointegration test and the wavelet theory. This new testing design has several advantages over the existing cointegration tests. First, it is a fully nonparametric test and does not necessitate the estimation of a parametric model as in Johansen (1988), or requires the selection of an optimal bandwidth as in Phillips and Ouliaris (1988). Second, the asymptotic distribution of the test is nuisance and tuning parameter free, and is invariant to the short run dynamics of the data. Third, the developed method can be applicable to a wide range of fractionally integrated series as well as standard $I(1)$ variables. Fourth, the frequency domain nature of the test give us the opportunity to focus on low frequency fluctuations and approximately remove any the problematic short run dynamics. Since the long run (co)movements appear in low frequency fluctuations, my method can easily capture the cointegration relations. Finally, by utilizing wavelets in cointegration framework, I propose a multivariate version of the wavelet based bootsrapping method particularly known as *wavestrapping*. This method is also nonparametric and very successful in removing size distortions appearing in cointegration tests.

The test proposed in this paper has good size and power features in small samples. Nevertheless, in general, cointegration tests are negatively affected by complicated short run dynamics in the data. An important cause of such dynamics is the highly negative MA roots in the innovation structure of stochastic trends. This, in fact, has been a widely recognized problem in the unit root testing literature. Since testing for cointegration is a multivariate generalization of unit root testing, it is not surprising that the problems such as these appear in the cointegration literature as well. For instance, Mallory and Lence (2012) claims that if a negative MA error structure is present in the data, using the standard asymptotic critical values for Johansen's (1988) cointegration test cause severe overrejection of the true null hypothesis. Moreover, from the simulation exercises we observe Nielsen's VR test (2010) also suffers severe size distortions in the aforementioned case. The wavelet based test can partially remove these size distortions. This feature can be attributed to the ability of wavelets to approximately filter out the short run dynamics in the observed data and the stochastic trends as well.

In order to remove the remaining size distortions, I design a wavelet based bootstrapping routine for my test without forsaking its non-parametric nature. In particular, Trokić (2016) showed how the wavelet based bootstrapping can be used to effectively reduce the size distortions in the case of unit root tests. Here, I develop a new method in a multivariate setting by using a similar wavelet based bootstrapping algorithm. This method is conducted via a wavelet filtering technique, namely the Discrete Wavelet Transform (DWT). To the best of my knowledge, this is the first paper which implements wavestrapping in a multivariate cointegration setup. Furthermore, I generalize the wavelet based bootstrapping approach to general fractionally integrated processes.

The rest of the article is organized as it follows: Section 2 reviews the DWT. In Section 3 I propose the new wavelet based variance ratio cointegration test. Section 4 is devoted to wavestrapping. Section 5 illustrates the small sample properties of the proposed test under different scenarios. In Section 6 I demonstrate the results of the empirical application of the proposed methods. The proof of the theorems and the tables for the simulation results can be found in the appendix. Throughout the paper, the following notation is used: $\xrightarrow{D}$ denotes weak convergence in distribution, $\xrightarrow{p}$ expresses convergence in probability, $\lfloor x \rfloor$ indicates the closest integer to x. Further, all the limits in this paper are taken as the sample size goes to ∞.

2 Wavelet Transform

In related recent studies, the use of wavelets in unit root and cointegration analysis has attracted some attention. In particular, Fan and Gencay (2010) develop a wavelet based unit root test. In their paper, the authors discuss that wavelets operate in both time and the frequency domain and adapt themselves to capture the nonstationarity features of variables across a wide range of frequencies (Fan and Gencay, 2010). This makes the wavelet transform a proper instrument for nonstationary time series and also cointegration analysis. Accordingly, for the construction of the new cointegration test, I utilize these tools, which are introduced below.

A wavelet, $\psi(t)$, is defined as a real-valued wave-like function fluctuating in a finite domain with the following properties:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \quad and \quad \int_{-\infty}^{\infty} \psi(t)^2 dt = 1$$

These two properties implies, a wavelet function must take non-zero value in a finite time period, but all the departures from zero should be cancelled out (Gencay, et al., 2000). By using a wavelet function, one can construct the continuous time wavelet transform (CWT) of a time series x_t as follows:

$$W(u,s) = \int_{-\infty}^{\infty} x_t \psi_{u,s}(t)dt$$

where $\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$ is translated by u and dilated by s. Note that u is the location and s is the scale parameter and $W(u,s)$ is called as wavelet coefficients. The parameter s, which take values in the $\mathbb{R}$, allows wavelets to work under different resolutions or in other words different frequencies. However, it is almost impossible to analyse all wavelet coefficients for different scales. Furthermore, the CWT creates a redundant transformation for time series data. Hence, the CWT is not an appropriate tool in my analysis. Nevertheless, the Discrete wavelet transformation (DWT), which shares the fundamental properties of the CWT, provides a non-redundant decomposition with finite number of scales. Consequently, the DWT is more suitable instrument for my study.

We can define the DWT with two separate filters. First, let $h = (h_0, h_1, ..., h_{L-1})$ denote a discrete wavelet (or high pass) filter with the finite length L. h_l corresponds to a filter coefficient for all $l = 0, ..., L-1$. The high pass filter h_l sums to 0, $\sum_{l=0}^{L-1} h_l = 0$ and it has unit energy, $\sum_{l=0}^{L-1} h_l^2 = 1$ like the CWT filters. These properties imply that the discrete wavelet filters fluctuate for a short period and fade out. As a result, wavelet filters does not treat all data points in the same way, unlike the Fourier transform. This feature is very useful to analyse nonstationary time series processes since the properties of nonstationary processes may alter through time.

In the DWT, we also have an additional filter g (low pass filter) which complements the high pass filter. The low pass filter g can be obtained by the quadrature mirror relationship[1]. Unlike the high pass filter, g sums to $\sqrt{2}$, $\sum_{l=0}^{L-1} g_l = \sqrt{2}$, but it has unit energy $\sum_{l=0}^{L-1} g_l^2 = 1$.

Convolving the data with these two filters, we decompose the time series process into its high frequency and low frequency components. Let $\{x_t\}_{t=1}^{T}$ be the observed time series process with dyadic length $T = 2^J$ for some integer J. Then, we define the matrix of the DWT coefficients $\mathcal{W}^L = \left[\mathbf{W}_1^L, \mathbf{W}_2^L, ..., \mathbf{W}_J^L, \mathbf{V}_J^L\right]'$, where for $j = 1, 2, ..., J$ $\mathbf{W}_j^L$ is the vector of j-th level *wavelet coefficients* and $\mathbf{V}_J^L$ is the vector of J-th level *scaling (approximation) coefficients*[2]. In this representation, the vector of the approximation coefficients $\mathbf{V}_J^L$ is associated with the fluctuations of x_t on the scale 2^J and the vector of the wavelet coefficients $\mathbf{W}_j^L$ is associated with the changes on the scale 2^{j-1}. Remark that, the scale is inversely proportional with frequency. As a result, $\mathbf{V}_J^L$ corresponds to the lowest frequency and $\mathbf{W}_1^L$ corresponds to the highest frequency components of the transformed series. Moreover, the approximation coefficient $\mathbf{V}_J^L$ has length of $T/2^J$ and $\mathbf{W}_j^L$ has length of $T/2^{j-1}$ for each $j = 1, 2..., J$.

In practice, the wavelet and the approximation coefficients can be obtained by the pyramid algorithm which is firstly proposed by Mallat (1989). This algorithm starts by filtering the observed series with the high pass ($\mathbf{h}$) and the low pass filters ($\mathbf{g}$) to obtain the first level wavelet and scaling coefficients:

$$V_{1,t}^L = \sum_{l=0}^{L-1} g_l x_{2t-l \bmod T} \qquad \textit{and} \qquad W_{1,t}^L = \sum_{l=0}^{L-1} h_l x_{2t-l \bmod T} \qquad \textit{for all } t = 1, 2, ..., T \tag{1}$$

where the filtering is carried out by the convolution of the observed series and the filters. Let the vectors $\mathbf{W}_1^L = \left\{W_{1,t}^L\right\}_{t=1}^{T/2}$ and $\mathbf{V}_1^L = \left\{V_{1,t}^L\right\}_{t=1}^{T/2}$ denote the first level wavelet and scaling coefficients respectively. Then, the matrix $[\mathbf{W}_1^L, \mathbf{V}_1^L]'$ constitutes the first level DWT of the time series x_t. For the levels $j = 2, ..., J$, we can obtain the j-th level coefficients by simply filtering the scaling coefficient of the level $j-1$:

$$V_{j,t}^L = \sum_{l=0}^{L-1} g_l V_{j-1,2t-l \bmod T}^L \qquad \textit{and} \qquad W_{j,t}^L = \sum_{l=0}^{L-1} h_l V_{j-1,2t-l \bmod T}^L$$

In the construction of my test statistic, I only use the first level DWT of the observed time series processes[3]. Although Trokić (2016) considers higher level transformations, it is clear from his findings that the first level DWT generates best results by means of power while the higher level DWT has slight size improvements in testing. This result is in accordance with the fact that in each application of the high pass and low pass filters, the half of the sample size of the previous level coefficients is lost.

As I mentioned earlier, I propose a wavelet based bootstrapping routine. In this routine, I utilize the higher level transformations than the level 1. The higher level wavelet coefficients provides more components to resample in bootstrapping. Furthermore, I use a multisignal version of the DWT in wavestrapping. In this version, I transform more than one series simultaneously. Although it seems this is not different than applying the DWT to individual series and stacking them in matrix form, the multisignal DWT help us preserving the cross-correlation structure between the resampled series and the original series. Further details will be given in subsequent sections.

The next section will combine the Variance Ratio cointegration testing with the wavelet theory to build up a powerful nonparametric cointegration test.

3 Wavelet Variance Ratio Cointegration Test

To construct my cointegration test, I first define the concepts of fractional integration and cointegration. For this part, I mostly follow Nielsen (2010).

[1]The quadrature mirror relationship can be characterized by: $g_l = (-1)^{l+1} h_{L-1-l}$ for $l = 0, ..., L-1$ (Fan and Gencay, 2010).

[2]Throughout the paper z^L notation indicates that z is an object associated with the wavelet decomposition with filter length L. L does not refer to power operator.

[3]In the lieterature, there are other variants of wavelet transformation such as the maximum overlap discrete wavelet transform and the discrete wavelet packet transform. However, in our analysis, the DWT gives the best results according to the simulation exercises. Consequently, we skip the explanation of these techniques in this paper.

Definition 1. The p vector time series Y_t is fractionally integrated of order d, denoted by $Y_t \in I(d)$ if

$$Y_t = \triangle_+^{-d} v_t,\ t = 1, 2, 3, ... \tag{2}$$

where v_t has a continuous spectral density matrix that is bounded positive semi-definite, and bounded away from the zero matrix at all frequencies, and the fractional integration operator $\triangle_+^{-d}$ can be defined as:

$$\triangle_+^{-d} y_t = \sum_{j=0}^{t} \pi_j(d) y_{t-j} \tag{3}$$

where $\pi_j(d) = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$ is the fractional binomial coefficient (Sowell, 1990) and only the past values of y_t with a positive index enter the integration.

With this specification, the right hand side of (3) is always well defined for all values of d (Nielsen, 2010). Note that although the series in (2) are never stationary for any $d \neq 0$, they are asymptotically stationary for $d < 1/2$. As in Nielsen (2010), I also use the term stationary for this case. In addition, if $d = 1$, Y_t becomes a standard unit root process, but the negative indexed history does not have any impact on the process.

Definition 2. The p vector $Y_t \in I(d)$ is said to be cointegrated with the rank r if there exist a full rank $p \times r$ matrix β such that $\beta' Y_t \in I(d - b)$ for $b > 0$ where $0 \leq r \leq p - 1$. Further, I assume $d - b < 1/2 < d$.

This definition generalizes the concept of cointegration to the fractional systems. Note that if we choose $d = b = 1$, the above characterization collapses to the standard $I(1)$-$I(0)$ cointegration framework. The representation in Definition 2, therefore, gives us a chance to deal with many different types of integrated time series which we encounter in the economics and finance literature. Furthermore, remark that there is an extra assumption on the fractional order of the cointegrating residuals which is restricted to be less than 1/2. This implies that the cointegrating residuals are stationary for any value of d. This technical assumption is needed for the proofs of the asymptotic theory established in the paper. Still, it is not too restrictive to impose such condition which says that the equilibrium error is stationary. Nonetheless, in simulations I show that when $d - b > 1/2$, my test still generates power against the null hypothesis. Additionally, the above assumption implies that we can have at most $p-1$ cointegrating relations among p time series. When $r = p$, we have all variables in Y_t are stationary.

To connect the above definitions of the fractional integration and the cointegration with testing and asymptotic theory, Nielsen (2010) suggests the following assumption:

Assumption 1. (Assumption of Cointegration) Y_t is the p vector of the observable variables which are $I(d)$. There exists a full rank orthonormal $p \times p$ matrix $R = [R_{p-r}, R_r]$, where R_{p-r} has $(p - r)$ columns and R_r has r columns with $0 \leq r \leq p - 1$ such that

$$R' Y_t = \begin{bmatrix} \triangle_+^{-d} I_{p-r} & 0_{(p-r)\times r} \\ 0_{(p-r)\times(p-r)} & \triangle_+^{-(d-b)} I_r \end{bmatrix} u_t \tag{4}$$

where I_m is an m-dimensional identity matrix and u_t is generated by a linear stationary process $u_t = \Psi(L)\epsilon_t = \sum_{k=0}^{\infty} \Psi_k \epsilon_{t-k}\ \forall\ t = 1, 2, ...$ The $p \times p$ coefficient matrices Ψ_k satisfies $\sum_{k=0}^{\infty} k^{1/2} \|\Psi_k\| < \infty$, $rank(\Psi(1)) = p$ if $r = 0$, $rank(\Psi(1)) \geq p - r + 1$ and $rank(\Psi_{11}(1)) = p - r$ if $r \geq 1$, where $\Psi_{11}(1)$ is the upper left $(p-r) \times (p-r)$ block of $\Psi(1) = \sum_{k=0}^{\infty} \Psi_k$; the remaining blocks being $\Psi_{12}(1)$ and $\Psi_{21}(1)$, and $\Psi_{22}(1)$. Finally ϵ_t are i.i.d with $\mathbb{E}[\epsilon_t] = 0$, $\mathbb{E}[\epsilon_t \epsilon_t'] = I_p$, $\mathbb{E}\, \|\epsilon_t\|^q < \infty$ for some $q > max(4, 2/(2d - 1))$.

This is the driving assumption of the Variance Ratio (VR) test of Nielsen (2010) and also of my test. Essentially, this assumption states that when $r \geq 1$, Y_t is cointegrated. On the other hand, when $r = 0$, all variables in Y_t follow distinct $I(d)$ processes which do not constitute any long run equilibrium. More importantly, equation (4) depicts a factorization for the cointegrated system. The orthonormal matrix R factorizes the observed times series Y_t into two components which asymptotically behave very differently from each other. The first component can be interpreted as a $(p - r)$-dimensional non-stationary factors. These factors are not cointegrated with each other but they construct the base of the long run equilibrium among the observed variables in Y_t. As a result, these factors can be linked to the common stochastic trend representations of cointegrated variables (see Stock and Watson (1988)). On the other hand, the second component of the factorization represents the stationary component of the equilibrium. These dynamics are associated with the cointegrating residuals or the equilibrium error which is illustrated in Definition 2. With these two separate set of processes, the characterization of the cointegration system is completed . Moreover, notice that when $p > r > 0$, the $(p-r)$-dimensional non-stationary factors become the common stochastic trends. However, when $r = 0$, we cannot claim that they are common across the observable variables. The other important outcome is that when $r = 0$, there is no stationary components in the system defined above.

Using this assumption, I construct a wavelet based cointegration test for fractionally integrated time series. In my analysis, I consider the type II fractionally integrated processes with the deterministic components. These components are restricted to mean and time trend for simplicity. The following assumption depicts the characteristics of the observed time series processes with these deterministic terms:

Assumption 2. The observed time series X_t is generated by:

$$X_t = \alpha'\delta_t + Y_t$$

where for $j = 0$ $\delta_t = 0$, for $j = 1$ $\delta_t = 1$ and for $j = 2$ $\delta_t = [1, t]'$. Further, $Y_t \in I(d)$ is the pure stochastic component.

In order to remove the deterministic components from the observed time series, we apply a OLS demeaning or detrending procedure. Hence, I use the residuals from the regression of the observed time series X_t on δ_t. The residuals are given by $\hat{Y}_t = Y_t - (\hat{\alpha} - \alpha)'\delta_t$.

Now applying the low pass filter (with length L) to each element of the time series $\hat{Y}_t$, we obtain $\hat{V}^L_{1,t}$. For analytic purposes, we use the compactly supported Daubechies class wavelet filters and their variants (see Daubechies (1988) for the further discussion). Here, we first define $V^L_{1,t} = \sum_{l=0}^{L-1} g_l Y_{2t-l \bmod T}$ which is the first level wavelet coefficient of the stochastic component Y_t. Additionally, if we use the demeaned or the detrended series in my analysis, we can set $\hat{V}^L_{1,t} = \sum_{l=0}^{L-1} g_l \hat{Y}_{2t-l \bmod T}$. Accordingly, this object has the following form:

$$\hat{V}^L_{1,t} = \sum_{l=0}^{L-1} g_l Y_{2t-l \bmod T} - (\hat{\alpha} - \alpha)' \sum_{l=0}^{L-1} g_l \delta_{2t-l \bmod T}$$

Further, we apply the fractional integration operator to $\hat{V}^L_{1,t}$ to obtain:

$$\tilde{\hat{V}}^L_{1,t} = \Delta^{d_1}_+ \hat{V}^L_{1,t}$$

Now we can define the objects $A^L_T = \sum_{t=1}^{T/2} \hat{V}^L_{1,t} \hat{V}^{L\prime}_{1,t}$ and $B^L_T = \sum_{t=1}^{T/2} \tilde{\hat{V}}^L_{1,t} \tilde{\hat{V}}^{L\prime}_{1,t}$. Note that A^L_T is the variance-covariance matrix of $\hat{V}^L_{1,t}$ and B^L_T is the variance covariance matrix of $\tilde{\hat{V}}^L_{1,t}$.

We are interested in testing the number of cointegrating relations for the observed series. For this purpose, we construct the null hypothesis as $H_0 : r = r_0$. Now, We can define the test statistic for this null hypothesis as $\Lambda^L_{p,r_0}(d_1) = T_1^{2d_1} \sum_{i=1}^{p-r_0} \lambda^L_i$ where λ^L_is are ordered eigenvalues of the ratio of two variance covariance matrices

$$A^L_T (B^L_T)^{-1} \tag{5}$$

The following theorem summarizes the main result of the paper.

Theorem 1. Suppose that Assumptions 1 and 2 holds. For $d_1 > 0$ and $r_0 = 0, 1, 2, ..., p - 1$

$$\Lambda^L_{p,r_0}(d_1) \xrightarrow{D} U_{p-r_0}(d, d_1)$$

$$U^L_{p-r_0}(d, d_1) = tr\left\{ \int_0^1 B^{p-r_0}_{j,d}(s) B^{p-r_0}_{j,d}(s)' ds \left(\int_0^1 B^{p-r_0}_{j,d,d_1}(s) B^{p-r_0}_{j,d,d_1}(s)' ds \right)^{-1} \right\}$$

as $T \to \infty$ where $B^{p-r_0}_{j,d}(s)$ is $p - r_0$ dimensional vector of standard ($j = 0$) or demeaned ($j = 1$) or detrended ($j = 2$) independent Fractional Brownian motions with fractional order d and $B^{p-r_0}_{j,d,d_1}(s)$ is $p - r_0$ vector of independent Fractional Brownian motions with fractional order $d + d_1$ for all $j = 0, 1, 2$. These Fractional

Brownian motions can be defined as:

$$
\begin{aligned}
B_{0,d}^{p-r_0}(s) &= W_d^{p-r_0}(s) \\
B_{j,d}^{p-r_0}(s) &= \left[W_d^{p-r_0}(s) - \left(\int_0^1 W_d^{p-r_0}(s) D_j(s)' ds \right) \right. \\
&\quad \left. \times \left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} D_j(s) \right] \; for\; j = 1,2 \qquad (6) \\
B_{0,d,d_1}^{p-r_0}(s) &= W_{d+d_1}^{p-r_0}(s) \\
B_{j,d,d_1}^{p-r_0}(s) &= \left[W_{d+d_1}^{p-r_0}(s) - \left(\int_0^1 W_d^{p-r_0}(s) D_j(s)' ds \right) \right. \\
&\quad \left. \times \left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} \int_0^s \frac{(s-r)^{d_1-1}}{\Gamma(d_1)} D_j(s) ds \right] \; for\; j = 1,2 \qquad (7)
\end{aligned}
$$

where $D_j(s) = 1$ if $j = 1$, and $D_j(s) = [1, s]'$ if $j = 2$ for $0 \leq s \leq 1$. These processes have same form as in Nielsen (2010).

Theorem 1 indicates that the new wavelet based test statistic has the same asymptotic distribution as Nielsen's VR (2010) test. This new test also carries over the most important features of Nielsen's (2010) test such as being nuisance and tuning parameter free. The test statistic only depends on d and d_1 and so there are no short run dynamics in the limiting distribution[4]. Moreover, the filter length is asymptotically irrelevant, since it does not appear in $U_{p-r_0}(d, d_1)$.

Finally,an additional theorem for the asymptotic power properties of the proposed test is presented below:

Theorem 2. Under the assumptions of Theorem 1, the test that rejects the null $H_0 : \; r = r_0$ when $\Lambda_{p,r_0}^L(d_1) > CV_{\xi,p-r_0}(d, d_1)$, where $CV_{\xi,p-r_0}(d, d_1)$ is found from

$$P\left(U_{p-r_0}(d, d_1) > CV_{\xi,p-r_0}(d, d_1)\right) = \xi$$

This implies that the proposed test has the asymptotic size ξ and is consistent against the alternative $H_1 : \; r > r_0$

In Theorem 2, $CV_{\xi,p-r_0}(d, d_1)$ is the asymptotic critical value for the null hypothesis $H_0 : r = r_0$. This theorem is important for constructing a testing strategy to determine the cointegration rank as indicated in Nielsen (2010). Since the asymptotic power is against the alternative $H_1 : \; r > r_0$, we can adopt a recursive testing scheme. Starting from the null $H_0 : \; r = 0$, if H_0 is rejected, we continue with testing the null $H_0 : \; r = 1$. This continues until we cannot reject the new null hypothesis or $r = p$. The value of r_0 where we stop testing is the estimate for the cointegration rank.

4 Wavestrapping for Cointegration Test

The wavelet variance ratio test proposed in the previous section performs better than the standard variance ratio test in the presence of the negative MA innovations. However, my test still suffers size distortion, particularly for highly negative MA roots. In order to address this issue, I exploit the wavelet nature of the proposed test to further eliminate any remaining size distortions. A common practice in the literature of unit roots is to apply a sieve parametric bootstrap method, such as in Nielsen (2009). Nevertheless, such bootstrap methods rely on parametric techniques such as specifying the parameter governing the sieve length. Even though Nielsen (2009) proposes a non-parametric unit root test, he relies on a parametric bootstrap for fixing size distortion issues. This, of course, implies that the bootstrapped test no longer enjoys the advantages of being fully non-parametric, so this could be somewhat of a disadvantage. My purpose in this paper, is to construct a wavelet based non-parametric bootstrap method which does not rely on any parametric estimation of the cointegration system. Further, apart from reducing size distortions, this nonparametric wavestrappping method will be an alternative to parametric bootstrap for cointegration rank determination (see Cavaliere et al. (2012)). The method proposed here is similar to Trokić (2016) and Percival and Walden (2006)) with significant extensions.

As its name indicates, wavestrapping is a bootstrap like procedure in which resampling is done via the wavelet transform of the observed series. The main idea behind this procedure stems from the fact that the discrete

[4] d can be determined a proiri to the testing or it can be obtained by pretesting as suggested in Nielsen (2009) and Nielsen (2010).

wavelet transform approximately decorrelates long memory time series (Percival and Walden, 2006). After obtaining and resampling the decorrelated coefficients, one can obtain the resampled process via the reconstruction filter. Moreover, my method is structurally different from the existing bootstrapping routines for cointegration testing. While other techniques use regression based resampling algorithms, I apply the wavestrapping routine to the common stochastic factors. These factors can easily be obtained nonparametrically from the eigenvalue problem of equation (5). According to Nielsen (2010), under the null hypothesis H_0 $r = r_0$ if we define $\eta(p - r_0)$ as the $p \times (p - r_0)$ matrix in which each column is the eigen vector corresponding to the $(p - r_0)$ smallest eigenvalues, these eigen vectors are in the direction of the non-cointegrated component (Nielsen, 2010). Then, if we post-multiply $\eta(p - r_0)$ with Y we obtained the stochastic trends upto a linear transformation, say $Z = Y\eta(p - r_0)$. The number of the stochastic trends may be more than one. In this case, my routine becomes multivariate wavestrapping algortihm. The DWT based wavestrapping can be summarized below:

1. Fix the Monte Carlo replications as MC, the significance level as α and the number of the bootstrap replication as B.
2. Given the observed time series $Y_{T\times p}$ matrix with length $T = 2^M$, set the level of the DWT as $J_0 = M - 2$.
3. Consider the null hypothesis $H_0 : r = r_0$. Compute the WVR test statistic $\Lambda_{p,r_0}(d_1)$[5] with the compactly supported wavelet with the filter length L.
4. Calculate the $T \times (p - r_0)$ dimensional stochastic trends Z as $Z = Y\eta(p - r_0)$ where $\eta(p - r_0)$ is the matrix of the eigen vectors defined above.
5. Similar to the classical bootstrapping methods, we need to resample the time series under the null hypothesis in which Z consists of $p - r_0$ independent unit root processes[6]. First, obtain the $T \times (p - r_0)$ matrix $e = \Delta Z$ as the first difference Z.
6. Apply a J_0 level multisignal 1 dimensional DWT transformation to e columnwise to retrieve the wavelet decomposition $\mathbf{w} = \{\mathbf{V}_{J_0}, \mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{J_0}\}$.[7] Note that $\mathbf{V}_{J_0}$ is a $T/2^{J_0} \times (p - r_0)$ matrix, of which each column is corresponding to the ${J_0}^{\text{th}}$ level approximation coefficients of the column vectors of e. Similarly, for $j = 1, ..., J_0$ $\mathbf{W}_j$ is a $T/2^j \times (p - r_0)$ matrix, of which columns correspond to j^{th} level wavelet coefficient of the column vectors of e.
7. Apply a parallel resampling scheme to only high pass filter part of the decomposition[8]. For bootstrap iteration $b = 1, ..., B$, obtain the resampled wavelet decomposition

$$\mathbf{w}^{*(b)} = \left\{\mathbf{V}_{J_0}, \mathbf{W}_1^{*(b)}, \mathbf{W}_2^{*(b)}, ..., \mathbf{W}_{J_0}^{*(b)}\right\}$$

   by resampling each $\mathbf{W}_j$ with replacement for $j = 1, 2, ..., J_0$. More clearly, let $\mathbf{W}_{j,t}$ be the t^{th} row of j^{th} level wavelet coefficient and $U_t^{*(b)}$ be an i.i.d sequence of discrete uniform distribution on $\{1, 2, ..., T/2^j\}$ [9]. Then $\mathbf{W}_{j,t}^{*(b)} = \mathbf{W}_{j,U_t^{*(b)}}$.
8. Finally employing the multisignal reconstruction filter to $\mathbf{w}^{*(b)}$, obtain $e^{*(b)}$ and $Z_t^{*(b)} = \sum_{i=1}^{t} e_i^{*(b)}$.
9. Compute the test statistic for the new wavestrapped sample $Z^{*(b)}$, say $\Lambda_{p,r_0}^{*(b)}(d_1)$.
10. Repeat the steps 7-9 B times to generate the wavestrapped empirical distribution for the test statistic.
11. Calculate the wavestrap p value from this empirical distribution:

$$p_m^* = \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}_{\left\{\Lambda_{p,r_0}(d_1) > \Lambda_{p,r_0}^{*(b)}(d_1)\right\}} \qquad \textit{for } m = 1, ..., MC$$

12. The wavestrap rejection probability for the Monte Carlo simulation can be computed as:

$$RP^* = \frac{1}{MC}\sum_{m=1}^{MC} \mathbb{1}_{\{p_m^* < \alpha\}}$$

[5]I drop the notation L, because of not complicating the presentation.

[6]In case of Fractional Cointegration, my method also works but it is slightly different. Instead of assuming we have $p - r$ unit roots, we assume we have $(p - r)$ $I(d)$ series under the null. Consequently instead of wavestrapping the first differenced series, we need to fractionally differentiate the observe series and apply the wavestrapping routine.

[7]This is equivalent to applying DWT each column of e.

[8]As stated in Tang et al. (2008), resampling the low pass filtered coefficients may distort the wavestrapping algorithm, since this component may not be uncorrelated

[9]The definition of $U_t^{*(b)}$ is very frequent in the bootstrapping literature (see Kilian (1998) and Cavaliere et al. (2012))

The steps 1-11 with $MC = 1$ describes the wavestrapping procedure for a particular sample. Step 12 is for the size distortion and the power evaluation exercises and it requires to compute $MC(B + 1)$ test statistic (Trokić, 2016). Trokić (2016) states that steps 1-10 of the DWT based resampling is a double bootstrap exercise which is relatively expansive to compute. However, Trokić (2016) indicates that one can use the Fast Double Bootstrap (wavestrap) procedure in which we only need to calculate $2MC$ number of the test statistic to inexpensively compute the rejection probability in step 12. In order to achieve this, first set $B = 1$ and estimate the RP^* as:

$$RP^* \simeq RP^*_{FDW} = \frac{1}{MC} \sum_{m=1}^{MC} \mathbb{1}_{\{\Lambda_{p,r_0}(d_1) > Q^*(1-\alpha)\}}$$

where $Q^*(1 - \alpha)$ is the $(1 - \alpha)^{\text{th}}$ quantile of the wavestrap test statistic $\Lambda^*_{p,r_0}(d_1)$.

Remark 1. In step 2, it is stated that the matrix of observations $Y_{T\times p}$ should have dyadic length. This may be restrictive for most cases in Economics and Finance. However, applying the signal extension techniques such as 'zero padding', 'symmetrization', 'smooth padding',... etc, one can still use the DWT transform for the variables with non-dyadic sample sizes (Strang and Nguyen, 1996). Moreover, software programs such as Matlab and R provides functions which computes the maximum number of possible decomposition level given the sample size T.

Remark 2. The so-called 'parallel bootstrap' is frequently applied in VAR literature (see Kilian (1998) and Cavaliere et al. (2012)). However, these methods require i.i.d innovations for resampling. In my method, practitioners do not need further action for the decorrelation of the innovations since wavelet transformation does this automatically without any parameter estimation. Additionally, the parallel bootstrap preserves the cross variable correlation. This feature save us from bothering about cross correlation between common stochastic components.

Remark 3. When $(p - r_0) = 1$, my wavestrapping routine is equivalent to Trokić's (2016) algorithm.

5 Small Sample Properties of the Wavelet VR Test and Wavestrapping

In this section, I evaluate the size and size-adjusted power performances of the proposed test, its wavestrapped version and Nielsen's (2010) variance ratio test via Monte Carlo simulations. Here, I restrict myself to the use of the compactly supported wavelets, namely the *Daubechies* wavelets with the length 2 and 4[10], and the Least Asymmetric wavelets (Symlets) with length 4, 8 and 16[11]. The notation used for these filters are given in Table 1. Furthermore, I consider the tests results for the null hypotheses $r = 0$, $r = 1$[12]. I initially set $(p - r) = 2$ and generate two independent unit roots as the stochastic trends. Using these trends, I obtain the observed time series. On the other hand, I both consider standard I(1)-I(0) cointegration and I(d)-I(d-b) fractional cointegration. However, in the fractional cointegration case, I set $d = 1$ for simplicity, but let the other crucial parameter b take non-integer values.

Table 1: The Wavelet Filters Used in the Analysis

Filter	Filter length
$Haar$	2
D_2	4
S_2	4
S_4	8
S_8	16

Let $\epsilon_{1,t}$, $e_{2,t}$ and $v_{2,t}$ be the independent and identically distributed standard normal draws. We obtain the stochastic trend $z_{1,t}$ as AR(1) unit root process. Applying the wavelet VR testing procedure defined in Section 3, we can find the value of the test statistic in each simulation. Also note that we reject the null when the test statistic is larger than the corresponding critical value. Consider now the DGPs for each test of interest:

[10] The *Daubechies* wavelet with filter length 2 is also known as "Haar"

[11] I also implement my method with other wavelets from *Daubechies* and Symlets, but by means of size and power, the best results are obtained by using those reported.

[12] I run the simulations for the null $H_0 : r = 2$, results are similar and can be provided upon request from the author

(a) $H_0: r = 0$

$$\begin{aligned} z_{1,t} &= z_{1,t-1} + \epsilon_{1,t} \\ \epsilon_{2,t} &= e_{2,t} + \theta e_{2,t-1} \\ z_{2,t} &= \rho_c z_{2,t-1} + \epsilon_{2,t} \\ y_{1,t} &= z_{1,t} \\ y_{2,t} &= z_{1,t} + z_{2,t} \end{aligned}$$

where $\rho_c = (1 - c/T)$. Note that if $c = 0$ then $y_{1,t}$ and $y_{2,t}$ are not cointegrated and so $r = 0$ holds with common stochastic trends $z_{1,t}$ and $z_{2,t}$. However, if $c > 0$ then $z_{2,t}$ is a stationary process and these two variables are cointegrated, thereby satisfying the alternative hypothesis. Moreover, MA(1) parameter of $\epsilon_{2,t}$, theta is crucial for checking the size distortions.

(b) $H_0: r = 1$

$$\begin{aligned} z_{1,t} &= z_{1,t-1} + \epsilon_{1,t} \\ \epsilon_{2,t} &= e_{2,t} + \theta e_{2,t-1} \\ z_{2,t} &= \rho_c z_{2,t-1} + \epsilon_{2,t} \\ y_{1,t} &= z_{1,t} \\ y_{2,t} &= z_{1,t} + v_{2,t} \\ y_{3,t} &= z_{1,t} + v_{2,t} + z_{2,t} \end{aligned}$$

Similar to the case above, if $c = 0$, $z_{2,t}$ will be nonstationary and we have only one cointegrating relation which is between $y_{1,t}$ and $y_{2,t}$. However, if $c > 0$ then $z_{2,t}$ will be stationary and we have further another cointegrating relation between $y_{1,t}$ and $y_{3,t}$. Consequently, $r = 2$. Further, θ has a simular role as in the DGP (a).

(c) $H_0: r = 1$

$$\begin{aligned} z_{1,t} &= z_{1,t-1} + \epsilon_{1,t} \\ z_{2,t} &= \rho_c z_{2,t-1} + e_{2,t} \\ y_{1,t} &= z_{1,t} \\ u_{2,t} &= v_{2,t} + \theta v_{2,t-1} \\ y_{2,t} &= z_{1,t} + u_{2,t} \\ y_{3,t} &= z_{1,t} + u_{2,t} + z_{2,t} \end{aligned}$$

where $\epsilon_{1,t}$, $e_{2,t}$ and $v_{2,t}$ are standard normal draw which is independent of each other. This DGP is very similar to (b) by means of cointegration structure: if $c = 0$, $z_{2,t}$ will be nonstationary and we have only one cointegrating relation which is between $y_{1,t}$ and $y_{2,t}$. However, if $c > 0$ then $z_{2,t}$ will be stationary and we have further another cointegrating relation between $y_{1,t}$ and $y_{3,t}$. Consequently, $r = 2$. Note, that the MA(1) coefficient θ only appears in the cointegrating residuals under both the null and the alternative.

(d) $H_0: r = 0$

$$\begin{aligned} z_{1,t} &= z_{1,t-1} + \epsilon_{1,t} \\ \epsilon_{2,t} &= e_{2,t} + \theta e_{2,t-1} \\ z_{2,t} &= \Delta_+^{b-1} \epsilon_{2,t} \\ y_{1,t} &= z_{1,t} \\ y_{2,t} &= z_{1,t} + z_{2,t} \end{aligned}$$

where b controls the fractional cointegration such that iff $b = 0$ then $y_{1,t}$ and $y_{2,t}$ are not cointegrated and so $r = 0$ holds with the stochastic trends $z_{1,t}$ and $z_{2,t}$. However, if $b > 1/2$[13] then $z_{2,t}$ is a stationary process and these two variables are cointegrated, so we are under the alternative hypothesis.

[13] I also consider the cases $b < 1/2$. Results are reported in the tables.

(e) $H_0 : r = 1$

$$
\begin{aligned}
z_{1,t} &= z_{1,t-1} + \epsilon_{1,t} \\
\epsilon_{2,t} &= e_{2,t} + \theta e_{2,t-1} \\
z_{2,t} &= \Delta_+^{b-1} \epsilon_{2,t} \\
y_{1,t} &= z_{1,t} \\
y_{2,t} &= z_{1,t} + v_{2,t} \\
y_{3,t} &= z_{1,t} + v_{2,t} + z_{2,t}
\end{aligned}
$$

As in the DGP (d), if $b = 0$, $z_{2,t}$ will be nonstationary and we have only one cointegrating relation which is between $y_{1,t}$ and $y_{2,t}$. However, if $b > 1/2$ then $z_{2,t}$ will be stationary and we have another cointegrating relation between $y_{1,t}$ and $y_{3,t}$ with fractionally integrated residuals. Consequently, $r = 2$.

Remark that we use the same critical values in all simulations since in all simulations we have $p - r$ stochastic trends under the null hypothesis. Critical values are also obtained by simulation[14]. After we find the critical values, we perform the power and size exercise for the cointegration rank $r = 0, 1$ and also consider $d_1 = \{0.1, 1\}$.

Consider the size performance of the cointegration tests under different scenarios. Table 3 summarizes the results for the null $H_0 : r = 0$ in standard I(1)-I(0) cointegration setup. From this table, we can observe that Nielsen's (2010) VR and the proposed wavelet VR tests with different wavelets are generating similar size performances, which is close to nominal size, when MA(1) coefficient is positive under different deterministic adjustment and also for $d_1 = 0.1$ and 1. However, the presence of the negative MA roots severely distorts the size performance in all of these cases, especially when we have deterministic component adjustment. When $\theta = -0.9$, the least size distortion is obtained by the S_2 and S_4 under any scenario. A similar pattern in Table 4 is also apparent for $H_0 : r = 1$ case. These findings imply that Symlet filters would be good choice if we want to reduce the size distortion in cointegration tests. To further remove remaining size distortions and obtain better small sample inference, we also conduct wavestrapping exercise for my tests. With the help of wavestrapping, we can achieve approximately 75% size reduction in *Haar* and D_2 filters based tests and 85% size reduction in the tests based symlet filters compared to FWVR test. Another important outcome from Tables 3 and 4 is that d_1 parameter drastically affect size performance under the negative MA root scenarios. Although $d_1 = 1$ causes slight undersizing in wavestrapping case, still the tests generated by this value have much more better size performance than the tests with $d_1 = 0.1$.

On the other hand, a natural question to ask is whether the presence of negative MA innovations in cointegrating residuals impact the size performance. In order to understand this, I consider the setup when $r = 1$ under the null hypothesis. Notice that we need at least one cointegrating relation under the null hypothesis. Table 5 summarizes the results for this cases where the data is generated according to the DGP (c). From this table, it is clear that the negative MA root presence does not affect the tests under the null hypothesis. However, this is an expected results because only stochastic trends are asymptotically important for Nielsen's (2010) and my test statistics.

I, next, consider the size-adjusted power features of the cointegration tests. For this exercise, I utilize the DGPs (a), (b), (d) and (e). While DGPs (a) and (b) corresponds to classical I(1)-I(0) cointegration framework (Tables 6-9), the DGPs (d) and (e) corresponds to Fractional cointegration case (Tables 10-13)[15].

First, take Classical I(1)-I(0) with $d_1 = 0.1$ case in to account. Table 6 displays that wavelet based tests can produce power under the alternative hypothesis with small sample size (T=128). Although Nielsen's (2010) tests seems to be more powerful when MA coefficient θ is positive, the negative values of θ engender more power for the wavelet based tests relative to Nielsen's (2010). Especially, D_2 and S_2 filters produce generally more power than other filters when they are used in the wavelet based test. However, the best results for wavestrapping are obtained by using *Haar* filter in almost any case. Another substantial result from Table 6 is that power drops for all test when we include deterministic component adjustment into the model. Furthermore, using $d_1 = 1$ leads to power loss in most cases except the scenario where $\theta = -0.9$. This power loss is eliminated by increasing the sample size. Table 7 clearly demonstrates that all wavelet based tests and Nielsen's (2010) have full power, but wavestrapping versions of my tests suffer slight power loss except the one with "Haar" filter.

[14] I simply simulate the functional of the fractional Brownian motions appeared in asymptotic distribution many times to obtain the critical values

[15] I dont include the size exercise for the fractional cointegration, since If the stochastic trends are still I(1), than the null hypothesis for fractional cointegration and standard I(1)-I(0) cointegration are identical.

Very similar results can be observed in Tables 8 and 9 for the null hypothesis $H_0 : r = 1$. Nonetheless, the wavelet tests with D_2, S_2, S_4 and S_8 gain power quickly with -0.9 MA(1) coefficient. This pattern is not persistent for other values of θ.

Consider Tables 10 to 13 for the size-adjusted power properties of the proposed methods in the fractional cointegration framework. The wavestrapped version of FWVR test with Haar filter shines in almost all cases. From other filters, D_2 and S_2 are doing fine, but the power results wavestrapped tests with S_4 and S_8 is not satisfactory. On the other hand, when $\theta = -0.9$ or -0.5, FWVR tests with $d_1 = 0.1$ and $d_1 = 1$ are producing similar power performance. However, this pattern does not last in positive values of θ and the tests with $d_1 = 0.1$ have the lead by means of the size-adjusted power performances. For the null hypothesis $r = 1$, results are similar for $d_1 = 0.1$, whereas the filters except $Haar$ are performing poorly in wavestrapping exercise when we consider demeaning or detrending. Finally, in the large samples, FWVR and its wavelet version is performing quite well by means of size adjusted power.

To sum up, it seems using $Haar$, D_2 or S_2 in the wavelet based cointegration tests and the wavestrapping routine yield satisfactory results by means size and size-adjusted power. These filters can be also mixed in testing and wavestrapping steps. For instance, one can use S_8 in FWVR test and $Haar$ in wavestrapping algorithm. For brevity of the paper, I just provide results with the case where same filter is utilized in the test statistic and wavestrapping method. Furthermore, $d_1 = 0.1$ seems to be optimal choice even though $d_1 = 1$ is doing better in a few case. However, $d_1 = 1$ usually fails to beat $d_1 = 0.1$ when we have positive MA(1) innovations. One can sacrifice the power gain obtained by the test constructed with $d_1 = 1$ to get a method working balanced in all situations.

6 Empirical Application

I apply my cointegration testing methodology to the European credit default swap (CDS) Rates data with different maturities. The credit default swaps are one of the most common type of credit derivatives, which can hedge the buyer against loses arising from a default (Longstaff et al., 2005). Thus, we can say that the CDS rate of a particular country indicates the default risk perception of the investors against that country. Accordingly, The main idea in using CDS rates in cointegration analysis is to check whether a particular maturity CDS rates of different countries within European Union co-move in the long run. I consider 8 countries, namely Austria, Belgium, France, Germany, Italy, Netherlands, Portugal and Spain.[16] For these countries, I collect one, two, three, five, seven and ten year CDS rates from Thomson Reuters. I use weekly data from 23 July 2008 to 23 November 2016[17]. This interval is the longest period for all countries and all maturities simultaneously. Furthermore, in the unreported unit root tests for these variables, I find all the CDS rates in my dataset follow a unit root.

In order to conduct cointegration test, I group the variables into 5 subsets of the data according to the maturities. That is, each subset contains 8 countries' CDS rates for a fix maturity. I separately employ Johansen's (1988) trace test(JT), FWVR test and its wavestrapped version to 5 datasets. I report the results in Table 2. In this table, we observe that according to JT test there is 5 cointegrating relations for one year CDS rate at 0.05 significance level. However, FWVR test finds 4 cointegrating relations at 0.05 significance level. Further, the wavestrapped FWVR test can only detect 1 cointegrating relation among the one year CDS rates of the different countries. One interesting result is that the wavestrapped FWVR test concludes no cointegration relation for the CDS rates with higher maturity than one year, while JT and $FWVR$ finds at least two long run relations among these variables.

For two year CDS rates JT and $FWVR$ tests reveal that cointegrating rank is 4. Whereas, the wavestrapped test can only find the presence of cointegraton at 0.10 significance level. On the other hand, when the maturity of the CDS rate increases, both JT and $FWVR$ test detect smaller number of cointegrating relations. For instance, for the cases with five, seven and 10 year CDS rates, JT test points the rank as 4, 3 and 2, respectively, but $FWVR$ test finds 3, 3, and 2 cointegrating relations for these cases, respectively.

To sum up, the empirical results demonstrate that $FWVR$ test tends to find less cointegrating relations than JT test in general for the CDS rate exercises. Nonetheless, both tests indicate the presence of the long run comovement among CDS rates with all maturities. On the other hand, from the wavestrapped test results for

[16] The broadest analysis can be done with these countries. Including the other EU countries decreases the length of the data set.

[17] The original data is daily, but, I convert it to weekly data as in Longstaff et al. (2005). The conversion is done by taking the mid of the week (Wednesdays) value of The CDS rates.

shorter maturity CDS rates, we can conclude that these variables co-move in the long run at least with one cointegrating relation. For the longer maturities, this result cannot be supported statistically. As a result, the wavestrapped version of my test draws quite different results than JT and $FWVR$ tests in CDS rate exercises. Nevertheless, this result is consistent with the findings of Beirne and Fratzscher (2013). The authors (Beirne and Fratzscher, 2013) claim that after the 2007-2008 sovereign debt crisis in Europe, there is sharp rise in the spread of sovereign yields across countries. Moreover some countries become more risky, thus CDS rates are mostly determined by country specific factors (Beirne and Fratzscher, 2013).

7 Conclusion

In this paper, I have proposed a non-parametric wavelet based cointegration testing method for fractionally integrated time series processes as well as the usual unit root variables. This method can be regarded as an alternative version of Nielsen's (2010) variance ratio test. In fact, these two tests share many common features. First, both tests are fully nonparametric and do not require the estimation of a regression model to remove short run dynamics. They also possess the same nuisance and tuning parameter free limiting distributions if one has a consistent estimate of the fractional integration parameter of the observed variables.

Although these two tests enjoy same properties asymptotically, they may exhibit different small sample characteristics in particular scenarios. Monte Carlo experiments indicate that when we compare power performances, my method can compete with the standard VR test in most cases and also beats it when we have the MA innovations with a negative root. On the other hand, if we study size distortions, the wavelet based test has superiority over the standard VR test under a wide range of specifications. Even so, the size distortions in the extreme cases, such as in the presence of the negative MA roots in the stochastic factors cannot be eliminated effectively by either of the tests. To handle such scenarios I propose a multivariate version of the wavestrapping procedure. Instead of wavestrapping the observed time series vector however, I wavestrap the stochastic factors which in fact carries all statistical features of the cointegration test in the Nielsen (2010) setup. The simulations demonstrate tremendous improvement in size distortion even in the extreme negative MA root instances. Further, The DWT based procedure exhibits local power under the different innovation structures.

Removing the size distortions in the particular cases is not the only benefit of the wavestrapping procedure. It also provides an alternative bootstrapping method for the determination of cointegration rank. To my knowledge this is this first attempt to implement a wavelet based bootstrapping algorithm in multivariate time series, particularly in a cointegration framework. Furthermore, the parametric bootstrapping methods which require the estimation of VECM model and the selection of the lag length parameter(Cavaliere et al. (2012) and Swensen (2006)), this new method remains fully nonparametric and tuning parameter free. In order to apply wavestrapping in cointegration framework, we only need to estimate the stochastic trends. These trends can be nonparametrically estimated from the eigenvalue problem of Nielsen's (2010) VR test explained in Section 3.

Finally, this paper has expanded the wavelet framework for the unit root testing to the multivariate models by providing the derivations for not only multiple time series processes but also for fractionally integrated systems. I believe that these findings may shed light on the properties of the wavelets transform of fractionally integrated variables in a multivariate setup.

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Appendix A: Proofs of the Theorems and the Lemmas

The proofs in this section have similarities with Nielsen (2010). However, they differ with the inclusion of wavelet theory, which makes the general proofs more challenging. Before the proof of the theorems, I need to illustrate some basic identities. First recall Assumption 1. In this assumption, we have the fundamental error term u_t which can be decomposed as:

$$u_t = \begin{bmatrix} u_t^{(1)} \\ u_t^{(2)} \end{bmatrix}$$

such that

$$R'Y_t = \begin{bmatrix} R'_{p-r}Y_t \\ R'_r Y_t \end{bmatrix} = \begin{bmatrix} \triangle_+^{-d} u_t^{(1)} \\ \triangle_+^{-(d-b)} u_t^{(2)} \end{bmatrix}$$

where, $u_t^{(1)}$ is the $(p-r)\times 1$ vector of the error terms that corresponds to the non-stationary element of the decomposition given in Assumption 1 and $u_t^{(2)}$ is the $r \times 1$ vector of the error term that correspond to the stationary element of this decomposition. Next, I define $Z_t = \triangle_+^{-d} u_t^{(1)}$ and $w_t = \triangle_+^{-(d-b)} u_t^{(2)}$[18]. Finally, we can write $R'_{p-r}Y_t = Z_t$ and $R'_r Y_t = w_t$.

Remark 4. In this section r and r_0 are used synonymously since the asymptotic theory is established under the null hypothesis of $H_0 : r = r_0$.

Now, I drop L only in $V_{1,t}^L$ for simplicity throughout the appendix. Then, I define the following objects as in Nielsen (2010):

$$F_T^L = \sum_{t=1}^{T/2} R'\hat{V}_{1,t}\hat{V}'_{1,t}R \tag{8}$$

$$H_T^L = \sum_{t=1}^{T/2} R'\tilde{\hat{V}}_{1,t}\tilde{\hat{V}}'_{1,t}R \tag{9}$$

where these objects can be decomposed as

$$F_T^L = \begin{bmatrix} \sum_{t=1}^{T/2} R'_{p-r}\hat{V}_{1,t}\hat{V}'_{1,t}R_{p-r} & \sum_{t=1}^{T/2} R'_{p-r}\hat{V}_{1,t}\hat{V}'_{1,t}R_r \\ \sum_{t=1}^{T/2} R'_r\hat{V}_{1,t}\hat{V}'_{1,t}R_{p-r} & \sum_{t=1}^{T/2} R'_r\hat{V}_{1,t}\hat{V}'_{1,t}R_r \end{bmatrix} = \begin{bmatrix} F_{11,T}^L & F_{12,T}^L \\ F_{12,T}^L & F_{22,T}^L \end{bmatrix}$$

$$H_T^L = \begin{bmatrix} \sum_{t=1}^{T/2} R'_{p-r}\tilde{\hat{V}}_{1,t}\tilde{\hat{V}}'_{1,t}R_{p-r} & \sum_{t=1}^{T/2} R'_{p-r}\tilde{\hat{V}}_{1,t}\tilde{\hat{V}}'_{1,t}R_r \\ \sum_{t=1}^{T/2} R'_r\tilde{\hat{V}}_{1,t}\tilde{\hat{V}}'_{1,t}R_{p-r} & \sum_{t=1}^{T/2} R'_r\tilde{\hat{V}}_{1,t}\tilde{\hat{V}}'_{1,t}R_r \end{bmatrix} = \begin{bmatrix} H_{11,T}^L & H_{12,T}^L \\ H_{12,T}^L & H_{22,T}^L \end{bmatrix}$$

Now, consider the following limits:

$$\begin{aligned} F_{11} &= 2^{2d}\Phi \int_0^1 B_{j,d}^{p-r}(s)B_{j,d}^{p-r}(s)'ds\Phi' \\ H_{11} &= 2^{2d}\Phi \int_0^1 B_{j,d+d1}^{p-r}(s)B_{j,d+d1}^{p-r}(s)'ds\Phi' \\ F_{22} &= var\left(\sum_{j=0}^{L-1} g_l \triangle_+^{-(d-b)} u_{2t-l}^{(2)}\right) \\ H_{22} &= var\left(\sum_{j=0}^{L-1} g_l \triangle_+^{-(d+d_1-b)} u_{2t-l}^{(2)}\right) \end{aligned}$$

where, $\Phi\Phi'$ is the long run variance of $u_t^{(1)}$ as given in Nielsen (2010). Then, the following lemma is from Nielsen (2010) and will be very crucial for my analysis.

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[18] Z_t notation is also used in the wavestrapping section as the vector of stochastic trends. In this section, Z_t can also be considered as the vector of stochastic trends, since it corresponds to nonstationary direction of Y_t.

Lemma 1. Nielsen (2010) Lemma 5. Let $X_{j,t} = \sum_{k=0}^{\infty} \tau_{j,k}\epsilon_{j,t-k}$ for $j = 1, 2$ be p_j dimensional stationary linear processes with $\sum_{k=0}^{\infty} \|\tau_{j,k}\| < \infty$ for $p_j \times p_j$ matrices $\tau_{j,k}$ and p_j dimensional i.i.d random variables $\epsilon_{j,t}$ with zero mean and finite fourth moments. Define the product moment $Q_{ijT}(u,v) = T^{-1}\sum_{t=1}^{T} \triangle_+^{-u} X_{i,t} \triangle_+^{-v} X'_{j,t}$,

(a) For $1/2 < v \leq u$ it holds that $T^{-u-v+1}Q_{ijT}(u,v) = O_p(1)$

(b) For $v < 1/2 < u$, $T^{-u+1/2}Q_{ijT}(u,v) \xrightarrow{p} 0$

(c) For $v = 1/2 \leq u$, $T^{-u-v+1}(logT)^{-\mathbb{1}_{\{u=1/2\}}}Q_{ijT}(u,v) = O_p(1)$

The proof can be found in Nielsen (2010) and I apply these results throughout the proof of the preceding lemma.

Lemma 2. Let the assumptions of Theorem 1 hold and let $\phi = \max(d - b + d_1, 1/2)$, as $T \to \infty$

(a) $T_1^{-2d} F_{11,T}^L = T_1^{-2d} \sum_{t=1}^{T/2} R'_{p-r} V_{1,t} V'_{1,t} R_{p-r} \xrightarrow{D} F_{11}$

(b) $T_1^{-1} F_{22,T}^L = T_1^{-2d} \sum_{t=1}^{T/2} R'_r V_{1,t} V'_{1,t} R_r \xrightarrow{D} F_{22}$

(c) $T_1^{-d-1/2} F_{12,T}^L = T_1^{-d-1/2} \sum_{t=1}^{T/2} R'_{p-r} V_{1,t} V'_{1,t} R_r \xrightarrow{p} 0$

(d) $T_1^{-2d-2d_1} H_{11,T}^L = T_1^{-2d} \sum_{t=1}^{T/2} R'_{p-r} \tilde{V}_{1,t} \tilde{V}'_{1,t} R_{p-r} \xrightarrow{D} H_{11}$

(e) $T_1^{-2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}} H_{22,T}^L = T_1^{-2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}} \sum_{t=1}^{T/2} R'_r \tilde{V}_{1,t} \tilde{V}'_{1,t} R_r \begin{cases} \xrightarrow{D} H_{22} & \text{if } d - b + d_1 < 1/2 \\ = O_p(1) & \text{if } d - b + d_1 \geq 1/2 \end{cases}$

(f) $T_1^{-d-d_1-\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}} H_{12,T}^L = T_1^{-d-d_1-\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}$
$\times \sum_{t=1}^{T/2} R'_{p-r} \tilde{V}_{1,t} \tilde{V}'_{1,t} R_r \begin{cases} \xrightarrow{p} 0 & \text{if } d - b + d_1 \leq 1/2 \\ = O_p(1) & \text{if } d - b + d_1 > 1/2 \end{cases}$

Lemma 2 is a generalization of Lemma 6 of Nielsen (2010) for the wavelet transformed series. Before I start the proof of this lemma, I define two new objects from the first level approximation coefficient. The first one is: $R_{p-r}V_{1,t} = \sum_{l=0}^{L-1} g_l Z_{2t-l} = G(L)\triangle_+^{-d} u_{2t}^{(1)}$ where $G(L) = g_0 + g_1 L + ... + g_{L-1}L^{L-1}$ [19]. Now, define $u_i^{(1)*} = G(L)u_i(1)$ for all i, then

$$R_{p-r}V_{1,t} = \sum_{i=1}^{2t} \pi_{2t-i}(d) u_{i-1}^{(1)*} \tag{10}$$

Similarly,

$$R_r V_{1,t} = \sum_{l=0}^{L-1} g_l w_{2t-l} = \sum_{i=1}^{2t} \pi_{2t-i}(d-b) u_{i-1}^{(2)*} \tag{11}$$

where $u_i^{(2)*} = G(L)u_i^{(2)}$ for all i. Further, the long run variance of the partial sum containing $u_{i-1}^{(1)*}$ is given as the $(p-r) \times (p-r)$ matrix

$$\Phi^*\Phi^{*'} = G(1)^2 \begin{bmatrix}\Psi_{11}(1) & \Psi_{12}(1)\end{bmatrix} \begin{bmatrix}\Psi_{11}(1) & \Psi_{12}(1)\end{bmatrix}' \tag{12}$$

This object is very similar to Φ, the variance covariance matrix given in the convergence represented equation (11) of Nielsen (2010). But the presence of $G(L)$ in $u_t^{(1)*}$ alters the result and we have $\Phi^* = G(1)\Phi$. However, we know that $G(1) = \sum_{l=0}^{L-1} g_l = \sqrt{2}$.

Proof of Lemma 2. (a) First assume $\delta_t = 0$, from equation (10), the partial sum processes for $R_{p-r}V_{1,t}$ can be written as:

$$\begin{aligned} T_1^{1/2-d} R_{p-r} V_{1,T_1}(s) &= T_1^{1/2-d} \sum_{i=1}^{2[T_1 s]-1} \pi_{2\lfloor T_1 s\rfloor - i}(d) u_{i-1}^{(1)*} \\ &= 2^{-1/2+d} T^{1/2-d} \sum_{i=1}^{\lfloor Ts\rfloor -1} \pi_{\lfloor Ts\rfloor - i}(d) u_{i-1}^{(1)*} \end{aligned} \tag{13}$$

[19] For brevity, I drop the mod T notation for the rest of the proofs.

where we can write $2[T_1 s] = [Ts]$. From Marinucci and Robinson (2000), $T_1^{1/2-d} R_{p-r} V_{1,T_1}(s) \xrightarrow{D} 2^{-1/2+d}\Phi^* W_d^{p-r}(s)$ as $T \to \infty$. The result follow from the FCLT for fractional Brownian motion of type II and the definition of the long run variance of $u_i^{(1)*}$ defined in equation (12). Finally, replacing $G(1) = \sqrt{2}$, we obtain $T_1^{1/2-d} R_{p-r} V_{1,T_1}(s) \xrightarrow{D} 2^d \Phi W_d^{p-r}(s)$. Now we can write:

$$T_1^{-2d} F_{11,T}^L = T_1^{-2d} \sum_{t=1}^{T/2} R'_{p-r} V_{1,t} V'_{1,t} R_{p-r}$$

Using the convergence result for $T_1^{1/2-d} R'_{p-r} V_{1,T_1}(r)$, we get

$$T_1^{-1} \sum_{t=1}^{T/2} T_1^{1-2d} R'_{p-r} V_{1,t} V'_{1,t} R_{p-r} = \int_0^1 T_1^{1-2d} R'_{p-r} V_{1,T_1}(s) V_{1,T_1}(s)' R_{p-r} ds$$

from the continuous mapping theorem we can deduce that

$$T_1^{-1} \sum_{t=1}^{T/2} T_1^{1-2d} R'_{p-r} V_{1,t} V'_{1,t} R_{p-r} \xrightarrow{D} 2^{2d} \Phi \int_0^1 W_d^{p-r}(s) W_d^{p-r}(s)' ds \Phi' = F_{11}$$

Also note that according to Lemma 1.a one can see that this product moment is $O_p(T^{2d})$. Furthermore, when $d = 1$ and $p = 1$ the same convergence rate is obtained by Fan and Gencay (2010) and Trokić (2016).

Now consider the cases $\delta_t = 1$ and $\delta_t = [1, t]'$ $(j \neq 0)$, then we need to use $\hat{Y}_t$ instead of Y_t. Remark that $\hat{V}_{1,t} = \sum_{l=0}^{L-1} g_l \hat{Y}_{2t-l}$. We can write $\hat{V}_{1,t} = V_{1,t} - (\hat{\alpha} - \alpha)' \sum_{l=0}^{L-1} g_l \delta_{2t-l}$. Now define $\tilde{\delta}_t = \sum_{l=0}^{L-1} g_l \delta_{2t-l}$. Then, clearly $\tilde{\delta}_t = \sqrt{2}$ if $j = 1$ and $\tilde{\delta}_t = [\sqrt{2}, 2\sqrt{2}t - \sum_{l=0}^{L-1} g_l l]'$ if $j = 2$. As in the previous part, we need to find the distribution of $T_1^{1/2-d} R'_{p-r} \hat{V}_{1,T_1}(s) = T_1^{1/2-d} R'_{p-r} V_{1,T_1}(s) - T_1^{1/2-d} R'_{p-r}(\hat{\alpha} - \alpha)' \tilde{\delta}_{\lfloor T_1 s \rfloor}$. I have already proven the convergence for the first part, now consider the second component:

$$T_1^{1/2-d} R'_{p-r}(\hat{\alpha} - \alpha)' \tilde{\delta}_{\lfloor Ts \rfloor} = T^{1/2-d} 2^{-1/2+d} R'_{p-r}(\hat{\alpha} - \alpha)' N_j(T)^{-1} N_j(T) \tilde{\delta}_{\lfloor T_1 s \rfloor}$$

where $N_j(T) = 1$ if $j = 1$ and $N_j(T) = diag([1, T^{-1}])$. I divide this object into two segment. For the first segment:

$$\begin{aligned} & T^{1/2-d} 2^{-1/2+d} R'_{p-r}(\hat{\alpha} - \alpha)' = \\ & 2^{-1/2+d} \left(T^{-1} \sum_{t=1}^{T} T^{1/2-d} R_{p-r} Y_t \delta'_t N_j(T) \right) \left(T^{-1} \sum_{t=1}^{T} N_j(T) \delta_t \delta'_t N_j(T) \right)^{-1} \xrightarrow{D} \\ & 2^{-1/2+d} \left(\Phi \int_0^1 W_d^{p-r}(s) D_j(s)' ds \right) \left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} \end{aligned} \tag{14}$$

where $D_j(s) = 1$ for $j = 1$ and $D_j(s) = [1, s]'$ for $j = 2$[20]. The second part $N_j(T)\tilde{\delta}_{\lfloor Ts \rfloor} \to \sqrt{2} D_j(s)$ for $0 \leq s \leq 1$. Notice that $\tilde{\delta}_{\lfloor T_1 s \rfloor} = [\sqrt{2} \quad 2\sqrt{2}\lfloor T_1 s \rfloor] = [\sqrt{2} \quad \sqrt{2}\lfloor Ts \rfloor]$ since $T_1 = T/2$. Accordingly, we can conclude:

$$\begin{aligned} T_1^{1/2-d} R'_{p-r} \hat{V}_{1,T_1}(s) & \xrightarrow{D} 2^d \Phi W_d^{p-r}(s) \\ & - 2^{-1/2+d} \left(\Phi \int_0^1 W_d^{p-r}(s) D_j(s)' ds \right) \times \left(\left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} 2^{1/2} D_j(s) \right) \\ & = 2^d \Phi W_d^{p-r}(s) - 2^d \left(\Phi \int_0^1 W_d^{p-r}(s) D_j(s)' ds \right) \times \left(\left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} D_j(s) \right) \\ & = 2^d \left[\Phi W_d^{p-r}(s) - \left(\Phi \int_0^1 W_d^{p-r}(s) D_j(s)' ds \right) \times \left(\int_0^1 D_j(s) D_j(s)' ds \right)^{-1} D_j(s) \right] \\ & = 2^d \Phi B_{j,d}^{p-r} \end{aligned}$$

[20]This is the same object as in equation (32) of Nielsen (2010). But in my case, we need to multiply this object with $2^{-1/2+d}$.

Finally, plugging this into the object $F^L_{11,T}$ we obtain:

$$T_1^{-1}\sum_{t=1}^{T/2} T_1^{1-2d} R'_{p-r}\hat{V}_{1,t}\hat{V}'_{1,t}R_{p-r} \xrightarrow{D} 2^{2d}\Phi\int_0^1 B^{p-r}_{j,d}(s)B^{p-r}_{j,d}(s)'ds\Phi' = F_{11} \quad \square$$

(b) Assume $\delta_t = 0$. First note that $R'_r Y_t = w_t$ for all t. Then we can expand the term $T_1^{-1}F^L_{22,T}$ as follows:

$$\begin{aligned}
T_1^{-1}F^L_{22,T} &= T_1^{-2d}\sum_{t=1}^{T/2} R'_r\left(\sum_{l=0}^{L-1} g_l Y_{2t-l}\right)\left(\sum_{l=0}^{L-1} g_l Y_{2t-l}\right)' R_r \qquad (15)\\
&= T_1^{-1}\sum_{t=1}^{T/2}\left(\sum_{l=0}^{L-1} g_l w_{2t-l}\right)\left(\sum_{l=0}^{L-1} g_l w_{2t-l}\right)'\\
&= T_1^{-1}\sum_{t=1}^{T/2}\left(\sum_{l=0}^{L-1}\Big(\sum_{j=0}^{L-1} g_l g_j w_{2t-l}w_{2t-j}\Big)\right)\\
&= T_1^{-1}\sum_{t=1}^{T/2}\Big(\sum_{j=0}^{L-1} g_l^2 w_{2t-l}^2\Big) + T_1^{-1}\sum_{t=1}^{T/2}\left(\sum_{l=0}^{L-1}\Big(\sum_{j\neq l}^{L-1} g_l g_j w_{2t-l}w_{2t-j}\Big)\right) \qquad (16)
\end{aligned}$$

Consider the first component in equation (16). w_{2t-1} is stationary and ergodic process when $d-b<1/2$ from equation (11), thus we can invoke the law of large number:

$$T_1^{-1}\sum_{t=1}^{T/2}\Big(\sum_{j=0}^{L-1} g_l^2 w_{2t-l}^2\Big) \xrightarrow{p} var\left(\sum_{j=0}^{L-1} g_l \triangle_+^{-(d-b)} u^{(2)}_{2t-l}\right)$$

But the second summation is $O_p(T_1^{2(d-b)})$ which has slower convergence rate then T_1. Thus, this term will vanish after scaling with T_1^{-1}. As a result $T_1^{-1}F^L_{22,T} \xrightarrow{D} F_{22}$.

Now assume $\delta_t \neq 0$ we can rewrite equation (15) as:

$$\begin{aligned}
T_1^{-1}F^L_{22,T} &= T_1^{-1}\sum_{t=1}^{T/2} R'_r(V_{1,t}-(\hat{\alpha}-\alpha)'\tilde{\delta}_t)(V_{1,t}-(\hat{\alpha}-\alpha)'\tilde{\delta}_t)'R_r\\
&= T_1^{-1}\sum_{t=1}^{T/2} R'_r V_{1,t}V'_{1,t}R_r - T_1^{-1}\sum_{t=1}^{T/2} R'_r V_{1,t}\tilde{\delta}'_t(\hat{\alpha}-\alpha)R_r \qquad (17)\\
&+ T_1^{-1}\sum_{t=1}^{T/2} R'_r(\hat{\alpha}-\alpha)'\tilde{\delta}_t\tilde{\delta}'_t(\hat{\alpha}-\alpha)R_r \qquad (18)
\end{aligned}$$

In equation (17), we have already found the distribution of the first summation. In order to find the asymptotic properties of the second summation in this equation, we divide it into two components:

$$(T_1^{-1}\sum_{t=1}^{T/2} R'_r V_{1,t}N_j(T)) \times (N_j(T)\tilde{\delta}'_t(\hat{\alpha}-\alpha)R_r) \qquad (19)$$

where the first component in equation (19) satisfies the following limiting behavior:

$$T_1^{-1}\sum_{t=1}^{T/2} R'_r V_{1,t}\tilde{\delta}_t N_j(T) = O_p(T^{d-b-1/2}) \qquad (20)$$

since $R'_r V_{1,t}$ should be scaled by $T^{1/2-(d-b)}$ for the convergence and $\tilde{\delta}_t N_j(T) = O(1)$ already. Further, the second part of equation (19) can be written as:

$$N_j(T)^{-1}(\hat{\alpha}-\alpha)R_r = \left(T^{-1}\sum_{t=1}^{T} N_j(T)\delta_t\delta'_t N_j(T)\right)^{-1}\left(T^{-1}\sum_{t=1}^{T} N_j(T)\delta_t Y'_t R_r\right) = O_p(T^{d-b-1/2}) \qquad (21)$$

from a similar argument as above. Multiplying the orders of these components, we can conclude that the second summation in (17) is $O_p(T^{2(d-b)-1})$. Since $2(d-b)-1<0$, this part converges to zero in probability.

Similarly, the final summation is given as:

$$T_1^{-1}\sum_{t=1}^{T/2} R_r'(\hat{\alpha}-\alpha)'\tilde{\delta}_t\tilde{\delta}_t'(\hat{\alpha}-\alpha)R_r = O_p(T^{2(d-b)-1}) \tag{22}$$

from the results in equation (21), then this object also vanishes as $T\to\infty$. Consequently, $T_1^{-1}F_{22,T}^L \xrightarrow{p} F_{22}$, this result is same as in the case where $j=0$. □

(c) For $j=0$, we have

$$\begin{aligned} T_1^{-d-1/2}F_{12,T}^L &= T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}'\left(\sum_{l=0}^{L-1} g_l Y_{2t-l}\right)\left(\sum_{l=0}^{L-1} g_l Y_{2t-l}\right)' R_r \\ &= T_1^{-d-1/2}\sum_{t=1}^{T/2}\left(\sum_{l=0}^{L-1} g_l Z_{2t-l}\right)\left(\sum_{l=0}^{L-1} g_l w_{2t-l}\right)' \end{aligned} \tag{23}$$

From equations (10) and (11) we know that $\left(\sum_{l=0}^{L-1} g_l Z_{2t-l}\right)$ and $\left(\sum_{l=0}^{L-1} g_l w_{2t-l}\right)$ are fractionally integrated processes with order d and $d-b$ respectively. Accordingly, we can apply Lemma 1.b. Then, the partial sum in equation (23) converges to 0 in probability. That is, $T_1^{-d-1/2}F_{12,T}^L \xrightarrow{p} 0$.

Now suppose $j=1,2$. We have,

$$\begin{aligned} T_1^{-d-1/2}F_{12,T}^L &= T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}'(V_{1,t}-(\hat{\alpha}-\alpha)'\tilde{\delta}_t)(V_{1,t}-(\hat{\alpha}-\alpha)'\tilde{\delta}_t)'R_r \\ &= T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}'V_{1,t}V_{1,t}R_r & (24) \\ &- T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}'V_{1,t}\tilde{\delta}_t'(\hat{\alpha}-\alpha)R_r & (25) \\ &- T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}(\hat{\alpha}-\alpha)'\tilde{\delta}_t V_{1,t}'R_r & (26) \\ &+ T_1^{-d-1/2}\sum_{t=1}^{T/2} R_{p-r}(\hat{\alpha}-\alpha)'\tilde{\delta}_t\tilde{\delta}_t'(\hat{\alpha}-\alpha)R_r & (27) \end{aligned}$$

We have already shown the convergence of the element in equation (24) above. We will show the remaining terms all converge to zero too.

First consider the object in equation (25); by Cauchy-Schwarz inequality this object is

$$O_p\left(\left(T_1^{2d}\sum_{t=1}^{T_1}\mathbb{E}\,\|R_{p-r}'V_{1,t}\|^2\right)^{1/2}\times\left(T_1^{-1}\sum_{t=1}^{T_1}\mathbb{E}\,\|\tilde{\delta}_t'(\hat{\alpha}-\alpha)R_r\|^2\right)^{1/2}\right)$$

First factor is $O_p(1)$ from Lemma 2 part a, and second factor is $o_p(1)$ from (22). Hence the multiplication is $o_p(1)$ which implies that the object in (25) converges to zero in probability. The object in (26) can be written as:

$$T_1^{-d+1/2}R_{p-r}(\hat{\alpha}-\alpha)N_j(T)T_1^{-1}\sum_{t=1}^{T/2}N_j(T)^{-1}\tilde{\delta}_t V_{1,t}'R_r$$

From equation (20), $T_1^{-1}\sum_{t=1}^{T/2}N_j(T)^{-1}\tilde{\delta}_t V_{1,t}'R_r = O_p(T^{d-b-1/2})$, and further $T_1^{-d+1/2}R_{p-r}(\hat{\alpha}-\alpha)N_j(T)$ is $O_p(1)$ from Lemma 1 part a. The multiplication becomes $O_p(T^{d-b-1/2})$ which indicates this object will converge to 0 also in probability.

Finally the last object is $O_p(T^{d-b-1/2})$ following the same arguments used for the convergences in equations (14) and (22). Since $d-b-1/2<0$, this object also converges to zero in probability. □

(d) Assume $j=0$. First, consider the partial sum process for fractionally transformed low pass filtered series. $\tilde{V}_{1,T_1}(t)$.

$$\tilde{V}_{1,T_1}(t) = T_1^{-d_1}\Delta_+^{-d_1}V_{1,T}(t) = T_1^{-d-d_1+1/2}\sum_{k=0}^{\lfloor T_1 t\rfloor-1}\pi_k(d_1)V_{1,\lfloor T_1 t\rfloor-k}$$
$$= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\pi_{\lfloor T_1 t\rfloor-k}(d_1)V_{1,k}$$

Now define $V_{1,k}=\sum\limits_{j=1}^{k}v_j$ with $\Delta V_{1,k}=v_k$

$$\begin{aligned}\tilde{V}_{1,T}(t) &= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\pi_{\lfloor T_1 t\rfloor-k}(d_1)V_{1,k}\\ &= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\pi_{\lfloor T_1 t\rfloor-k}(d_1)\sum_{j=1}^{k}v_j \qquad (28)\\ &= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\sum_{j=1}^{k}\pi_{\lfloor T_1 t\rfloor-k}(d_1)v_j\end{aligned}$$

$$\begin{aligned}&= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\pi_{\lfloor T_1 t\rfloor-k}(d_1+1)v_k \qquad (29)\\ &= T_1^{-d-d_1+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\frac{(\lfloor T_1 t\rfloor-k)^{d_1}}{\Gamma(d_1+1)}v_k\\ &= T_1^{-d+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\frac{\left(\frac{\lfloor T_1 t\rfloor-k}{T_1}\right)^{d_1}}{\Gamma(d_1+1)}v_k \qquad (30)\\ &= T_1^{-d+1/2}\sum_{k=1}^{\lfloor T_1 t\rfloor}\frac{(t-k/T_1)^{d_1}}{\Gamma(d_1+1)}v_k\\ &= \sum_{k=1}^{\lfloor T_1 t\rfloor}\frac{(t-k/T_1)^{d_1}}{\Gamma(d_1+1)}T_1^{-d+1/2}\triangle V_{1,k} \qquad (31)\end{aligned}$$

Here $T_1^{-d+1/2}\Delta V_{1,k}$ can be written as $\int_{(k-1)/T_1}^{k/T_1}dV_{1,T_1}(s)$ in the limit (see Phillips (1987)). Then,

$$\begin{aligned}\tilde{V}_{1,T_1}(t) &= \sum_{k=1}^{\lfloor T_1 t\rfloor}\frac{(t-k/T_1)^{d_1}}{\Gamma(d_1+1)}\int_{(k-1)/T_1}^{k/T_1}dV_{1,T_1}(s)\\ &= \sum_{k=1}^{\lfloor T_1 t\rfloor}\int_{(k-1)/T_1}^{k/T_1}\frac{(t-k/T_1)^{d_1}}{\Gamma(d_1+1)}dV_{1,T_1}(s) \qquad (32)\\ &\xrightarrow{D}\int_0^t\frac{(t-s)^{d_1}}{\Gamma(d_1+1)}dV_{1,T_1}(s) \qquad (33)\end{aligned}$$

But, $R'_{p-r}V_{1,T_1}(s)\xrightarrow{D}2^d\Phi W_d^{p-r}(s)$ from lemma 2 part a, then $R'_{p-r}\tilde{V}_{1,T_1}(s)\xrightarrow{D}2^d\Phi W_{d+d_1}^{p-r}(s)$ from CMT.

Assume $j = 1, 2$, we need to consider:

$$\begin{aligned}
T_1^{-d-d_1+1/2}\tilde{\tilde{V}}_{1,T_1}(s) &= T_1^{-d-d_1+1/2}\triangle_+^{-d_1}(R'_{p-r}V_{1,T_1}(s) - R'_{p-r}(\hat{\alpha}-\alpha)\tilde{\delta}_t) \\
&= T_1^{-d-d_1+1/2}\triangle_+^{-d_1}R'_{p-r}V_{1,T_1}(s) - T_1^{-d-d_1+1/2}\triangle_+^{-d_1}R'_{p-r}(\hat{\alpha}-\alpha)\tilde{\delta}_t \\
&= T_1^{-d-d_1+1/2}R'_{p-r}\tilde{V}_{1,T_1}(s) - T_1^{-d-d_1+1/2}\triangle_+^{-d_1}R'_{p-r}(\hat{\alpha}-\alpha)\tilde{\delta}_t
\end{aligned}$$

As we mentioned before, the first component will have the convergence:

$$T_1^{1/2-d-d_1}R_{p-r}\tilde{V}_{1,T_1}(s) \xrightarrow{p} 2^d \Phi W_{d+d_1}^{p-r}(s)$$

We will focus on the second object. Let $P_T = T_1^{-d-d_1+1/2}\triangle_+^{-d_1}R'_{p-r}(\hat{\alpha}-\alpha)'\tilde{\delta}_t$, consequently,

$$\begin{aligned}
P_T &= T_1^{-d-d_1+1/2}\sum_{k=1}^{t}\pi_{t-k}(d_1)R'_{p-r}(\hat{\alpha}-\alpha)\tilde{\delta}_k \\
&= \left(T_1^{1/2-d}R'_{p-r}(\hat{\alpha}-\alpha)'N_j(T)^{-1}\right) \\
&\times \left(T_1^{-d_1}\right)\sum_{k=1}^{t}\pi_{t-k}(d_1)N_j(T)\tilde{\delta}_k
\end{aligned} \tag{34}$$

The first factor converges $2^{-1/2+d}\left(\Phi\int_0^1 W_d^{p-r}(s)D_j(s)'ds\right)\left(\int_0^1 D_j(s)D_j(s)'ds\right)^{-1}$ from (14), the second factor is deterministic and converges to $\int_0^s \frac{(s-r)^{d_1-1}}{\Gamma(d_1)}2^{1/2}D_j(s)ds$ as in Nielsen (2010) and using the findings in Lemma 2.a. Finally, we obtain

$$\begin{aligned}
T_1^{-d-d_1+1/2}\tilde{\tilde{V}}_{1,T_1}(s) \xrightarrow{D} 2^d\Bigg[&\phi W_{d+d_1}^{p-r}(s) - \left(\Phi\int_0^1 W_d^{p-r}(s)D_j(s)'ds\right) \\
&\times\left(\int_0^1 D_j(s)D_j(s)'ds\right)^{-1}\int_0^s\frac{(s-r)^{d_1-1}}{\Gamma(d_1)}D_j(s)ds\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
B_{j,d,d_1}^{p-r}(s) = \Bigg[&\phi W_{d+d_1}^{p-r}(s) - \left(\Phi\int_0^1 W_d^{p-r}(s)D_j(s)'ds\right) \\
&\times\left(\int_0^1 D_j(s)D_j(s)'ds\right)^{-1}\int_0^s\frac{(s-r)^{d_1-1}}{\Gamma(d_1)}D_j(s)ds\Bigg]
\end{aligned}$$

Applying CMT as in Lemma 2.a, we achieve $T_1^{-2d-2d_1}H_{11,T}^L \xrightarrow{D} H_{11}$ □

(e) Again, say $j = 0$. When $d-b+d_1 < 1/2$ the proof exactly follows as in part b, but we need to replace the limit as H_{22} instead of F_{22} and the scaling with $T_1^{-2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}$. That is $T_1^{-2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}H_{22,T}^L \xrightarrow{D} H_{22}$. The result follows from the application of Cauchy-Schwarz inequality (see Nielsen (2010) for more details). When $d-b+d_1 > 1/2$, write

$$\begin{aligned}
T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}H_{22,T}^L &= T_1^{\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}R'_r\tilde{V}_{1,t}(\tilde{V}_{1,t})'R_r \\
&= T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}\Big(\sum_{j=0}^{L-1}g_l^2\tilde{w}_{2t-l}^2\Big) \qquad (35) \\
&+ T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}\Bigg(\sum_{l=0}^{L-1}\Big(\sum_{j\neq l}^{L-1}g_lg_j\tilde{w}_{2t-l}\tilde{w}_{2t-j}\Big)\Bigg) \qquad (36)
\end{aligned}$$

where $\tilde{w}_{2t-l} = \Delta_+^{-d_1}w_{2t-l}$. We can apply Lemma 1.a to all summations in equations (36) and (35) and see these summations are $O_p(1)$.

When $d-b+d_1=1/2$ we can apply Lemma 1.c to the summations in (36) and (35). As a result, all of these summations are $O_p(1)$.

Now assume $j=1,2$ and define $J_t=\triangle_+^{-d_1}\tilde{\delta}_t'(\hat{\alpha}-\alpha)R_r$

$$T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}H_{22,T}^L=T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}R_r'(\tilde{V}_{1,t}-J_t)(\tilde{V}_{1,t}-J_t)'R_r$$

$$=T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}R_r'\tilde{V}_{1,t}\tilde{V}_{1,t}'R_r \tag{37}$$

$$-T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}R_r'\tilde{V}_{1,t}J_t'R_r \tag{38}$$

$$-T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}R_r'J_t\tilde{V}_{1,t}'R_r \tag{39}$$

$$+T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}J_tJ_t \tag{40}$$

First note that $J_t=O_p(T^{d-b+d_1-1/2})$ from (34) if we plug R_r instead of R_{p-r}. Furthermore, the object in (40) is

$$O_p\left(T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\sum_{t=1}^{T/2}\mathbb{E}\left\|J_t\right\|^2\right)=O_p(1)$$

Next, for equations (38) and (39), we apply Cauchy-Schwarz inequality then these objects become

$$O_p\left(T_1^{2\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\big(\sum_{t=1}^{T/2}\mathbb{E}\left\|J_t\right\|^2\big)^{1/2}\big(\sum_{t=1}^{T/2}\mathbb{E}\left\|R_r'\tilde{V}_{1,t}\right\|^2\big)^{1/2}\right)=O_p(1)$$

Since these three objects are $O_p(1)$, the result does not change after including the deterministic terms in the model. □

(f) If $d-b+d_1<1/2$, the proof is similar as part (c). When $d-b+d_1\geq 1/2$.

$$T_1^{-d-d_1-\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}H_{12,T}^L=O_p(T_1^{-d-d_1-\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}\times\sum_{t=1}^{T/2}\mathbb{E}[R_{p-r}'\hat{\tilde{V}}_{1,t}\hat{\tilde{V}}_{1,t}'R_r])$$

now applying Cauchy-Schwarz inequality we have:

$$O_p\left(T_1^{-d-d_1-\phi}(\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}}H_{12,T}^L\right)\leq O_p\left(\left(T_1^{-2d-2d_1}\sum_{t=1}^{T/2}\mathbb{E}\left\|R_{p-r}'\tilde{V}_{1,t}\right\|^2\right)^{1/2}\times\left(T_1^{-2\phi}(\log(T_1))^{-2\mathbb{1}_{\{\phi=1/2\}}})\sum_{t=1}^{T/2}\mathbb{E}\left\|R_r'\tilde{V}_{1,t}\right\|^2\right)^{1/2}\right)$$

Now, the first part is $O_p(1)$ from Lemma 2.d. Additionally, the second part is $O_p((\log(T_1))^{-\mathbb{1}_{\{\phi=1/2\}}})$ which becomes $O_p(1)$ if $d-b+d_1>1/2$ and if $d-b+d_1=1/2$, this term will be $O_p(\log(T_1)^{-1})$ and it converges to 0 in probability. □

Proof of Theorem 1. Lemma 2 will be the main engine of the proof of Theorem 1. This proof will proceed similarly as the proof of Theorem 1 of Nielsen (2010). The difference is that instead of scaling the objects with functions of T, we scale them with $T_1=T/2$. Since we have the same convergence properties for the newly defined wavelet based moments, we will obtain the same asymptotic result as in Nielsen (2010)[21].

First note that my test statistic $\Lambda^L_{p,r}(d_1)$ is a function of the ordered eigenvalues of $A^L_T(B^L_T)^{-1}$. These are also the eigenvalues of the eigenvalue problem

$$|\lambda H^L_T - F^L_T| = 0 \tag{41}$$

where F^L_T and H^L_T are defined in equations (8) and (9) respectively. Pre and post multiplication of F^L_T and H^L_T with the orthonormal matrix R does not change the eigenvalue problem. Let,

$$\begin{aligned} M^L_T &= \begin{bmatrix} T_1^{-d-d_1-1/2+\phi} I_{p-r} & 0_{p-r\times r} \\ 0_{r\times p-r} & T_1^{-1/2} I_r \end{bmatrix} \quad and \\ \overline{F}^L &= \begin{bmatrix} 0_{p-r\times p-r} & 0_{p-r\times r} \\ 0_{r\times p-r} & F^L_{22} \end{bmatrix} \end{aligned} \tag{42}$$

If we pre and post multiply F^L_T and H^L_T with M^L_T and apply Lemma 2 we get:

$$M^L_T F^L_T M^L_T = \begin{bmatrix} T_1^{-2d-2d_1-1+2\phi} F^L_{11,T} & T_1^{-d-d_1-1+\phi} F^L_{12,T} \\ T_1^{-d-d_1-1+\phi} F^L_{12,T} & T_1^{-1} F^L_{22,T} \end{bmatrix} \xrightarrow{p} \overline{F}^L \tag{43}$$

$$T_1^{1-2\phi} M^L_T H^L_T M^L_T = \begin{bmatrix} T_1^{-2d-2d_1} H^L_{11T} & T_1^{-d-d_1-\phi} H^L_{12T} \\ T_1^{-d-d_1-\phi} H^L_{12T} & T_1^{-2\phi} H^L_{22,T} \end{bmatrix},$$

where $T_1^{-2d-2d_1} H^L_{11T} \xrightarrow{D} H^L_{11}$ which is symmetric and positive definite a.s (from Lemma 2.d), $T_1^{-d-d_1-\phi} H^L_{12T} = O_p(1)$ (from Lemma 2.f), $T_1^{-2\phi} H^L_{22,T} = O_p(1)$ (from Lemma 2.e) and F^L_{22} is deterministic, symmetric and positive definite.

Let $\rho = \lambda T_1^{2\phi-1}$. The ordered eigenvalues of (41) are the same as:

$$|\rho T_1^{1-2\phi} M^L_T H^L_T M^L_T - M^L_T F^L_T M^L_T| = 0$$

Now we define $\Upsilon(A, B)$ as the function that associates with a pair of matrices (A, B) and the ordered eigenvalues of the problem $|\rho A - B| = 0$ (Nielsen, 2010). Note that this function is continuous in its two matrix arguments and additionally we also have $(T_1^{1-2\phi} M^L_T H^L_T M^L_T, M^L_T F^L_T M^L_T)$ is tight, it follows from equation (43) that

$$\|\Upsilon(T_1^{1-2\phi} M^L_T H^L_T M^L_T, M^L_T F^L_T M^L_T) - \Upsilon((T_1^{1-2\phi} M^L_T H^L_T, \overline{F}^L)\| \xrightarrow{p} 0$$

The solutions to right hand side of convergence $\Upsilon((T_1^{1-2\phi} M^L_T H^L_T, \overline{F}^L)$ satisfy

$$\begin{aligned} 0 &= \left| \rho \begin{bmatrix} T_1^{-2d-2d_1} H^L_{11T} & T_1^{-d-d_1-\phi} H^L_{12T} \\ T_1^{-d-d_1-\phi} H^L_{12T} & T_1^{-2\phi} H^L_{22,T} \end{bmatrix} \right| - \begin{bmatrix} 0_{p-r\times p-r} & 0_{p-r\times r} \\ 0_{r\times p-r} & F^L_{22} \end{bmatrix} \\ &= |\rho T_1^{-2d-2d_1} H^L_{11T}||\rho \Big(T_1^{-2\phi} H^L_{22,T} - (T_1^{-d-d_1-\phi} H^L_{12T})' \\ &\times (T_1^{-2d-2d_1} H^L_{11T})^{-1} (H^L_{22,T} - (T_1^{-d-d_1-\phi} H^L_{12T}))\Big) - F^L_{22}| \end{aligned} \tag{44}$$

since $T_1^{-2d-2d_1} H^L_{11T}$ is symmetric and positive definite a.s.

From the first absolute value in equation (44), $p_j \xrightarrow{p} 0$ for $j = 1, ..., p-r$ as $T_1^{-2d-2d_1} H^L_{11T} \xrightarrow{D} H^L_{11}$ which is

[21] The remaining part of the proof follows closely the proof of Theorem 1 in Nielsen (2010).

symmetric and positive definite a.s. For the second absolute value in (44), consider $\nu = \rho^{-1}$ which solves

$$\begin{aligned}
0 &= |\Big(T_1^{-2\phi} H_{22,T}^L - (T_1^{-d-d_1-\phi} H_{12T}^L)' \\
&\times (T_1^{-2d-2d_1} H_{11T}^L)^{-1} (H_{22,T}^L - (T_1^{-d-d_1-\phi} H_{12T}^L))\Big) - \nu F_{22}^L| \\
&= |(F_{22}^L)^{-1/2} \Big(T_1^{-2\phi} H_{22,T}^L - (T_1^{-d-d_1-\phi} H_{12T}^L)' \\
&\times (T_1^{-2d-2d_1} H_{11T}^L)^{-1} (H_{22,T}^L - (T_1^{-d-d_1-\phi} H_{12T}^L))\Big) F_{22}^L)^{-1/2} - \nu I_r| \qquad (45)
\end{aligned}$$

where the second equality holds because F_{22}^L is symmetric and positive definite. We utilize CMT and have $(T_1^{-2d-2d_1} H_{11T}^L)^{-1} \xrightarrow{D} (H_{11}^L)^{-1}$ since H_{11}^L is positive definite a.s. As a result, $(T_1^{-2d-2d_1} H_{11T}^L)^{-1} = O_p(1)$, further $T_1^{-d-d_1-\phi} H_{12T}^L = O_p(1)$ and $T_1^{-2\phi} H_{22,T}^L = O_p(1)$. Then the solution of (45) satisfies $\nu_j = O_p(1)$ for $j = 1, ..., r$. This implies that $\rho_j = O_p(1)$ for $j = n-r+1, ..., n$ and consequently the r largest roots of (5) satisfy $\lambda_j^{-1} = O_p(T^{2\phi-1})$ for $j = n-r+1, ..., n$. Let

$$K_T^L = \begin{bmatrix} T_1^{-d} I_{p-r} & 0_{p-r \times r} \\ 0_{r \times p-r} & T_1^{-1/2} I_r \end{bmatrix}$$

such that, by Lemma 2.a-c

$$K_T^L F_T^L K_T^L = \begin{bmatrix} T_1^{-2d} F_{11,T}^L & T_1^{d-1/2} F_{12,T}^L \\ T_1^{d-1/2} F_{12,T}^L & T_1^{-1} F_{22,T}^L \end{bmatrix} \xrightarrow{D} \begin{bmatrix} F_{11}^L & 0_{p-r \times r} \\ 0_{r \times p-r} & F_{22}^L \end{bmatrix} \qquad (46)$$

and

$$\begin{aligned}
T_1^{-2d_1} K_T^L H_T^L K_T^L &= \begin{bmatrix} T_1^{-2d-2d_1} H_{11T}^L & T_1^{d-1/2-2d_1} H_{12T}^L \\ T_1^{d-1/2-2d_1} H_{12T}^L & T_1^{-1-2d_1} H_{22,T}^L \end{bmatrix} \\
&\xrightarrow{D} \begin{bmatrix} H_{11}^L & 0_{p-r \times r} \\ 0_{r \times p-r} & 0_{r \times r} \end{bmatrix} \qquad (47)
\end{aligned}$$

The ordered eigenvalues of (41) are the same as the eigenvalues of: $|\lambda K_T^L H_T^L K_T^L - K_T^L F_T^L K_T^L| = 0$ and setting $\mu = (T_1^{2d_1} \lambda)^{-1}$, these eigenvalues are same as those of

$$|T_1^{-2d_1} K_T^L H_T^L K_T^L - \mu K_T^L F_T^L K_T^L| = 0 \qquad (48)$$

Since the eigenvalues are continuous functions of the argument matrices, (46) and (47) indicates that the ordered eigenvalues of (48) converges in distribution to those of

$$\left| \begin{bmatrix} H_{11}^L & 0_{p-r \times r} \\ 0_{r \times p-r} & 0_{r \times r} \end{bmatrix} - \mu \begin{bmatrix} F_{11}^L & 0_{p-r \times r} \\ 0_{r \times p-r} & F_{22}^L \end{bmatrix} \right| = 0$$

This equation has r zero roots (since F_{22}^L is deterministic and positive definite) and $p-r$ a.s. positive roots (because both H_{11}^L and F_{11}^L are symmetric and positive definite a.s) given by the solutions of $|H_{11}^L - \mu F_{11}^L| = 0$.

In light of the above results, we obtain the scaled eigenvalues of the original eigenvalue problem $T_1^{2d_1} \lambda_1^L, ..., T_1^{2d_1} \lambda_{p-r}^L$ converge in distribution to the solutions of $|T_1^{2d_1} \lambda H_{11} - F_{11}| = 0$, then $T_1^{2d_1} \sum_{j=1}^{p-r} \lambda_j^L \xrightarrow{D} tr \left\{ F_{11} (H_{11})^{-1} \right\}$. Finally note that the trace operator will remove the Ψ terms from $F_{11}(H_{11})^{-1}$. □

The proof of Theorem 2 directly follows the proof of Theorem 2 of Nielsen (2010).

Proof of Theorem 2. In the proof of Theorem 1, it is shown that $(\lambda_{p-r+1}^L)^{-1}, ..., (\lambda_{p-r_0}^L)^{-1}$ are all $O_p(T^{2\phi-1})$. Hence, under the alternative, the test statistic $\Lambda_{p,r}^L(d_1) = T_1^{2d_1} \sum_{j=1}^{p-r_0} \lambda_j^L \geq T_1^{2d_1} \lambda_{p-r_0}^L$ and $(T_1^{2d_1} \lambda_{p-r_0}^L)^{-1} = O_p(T^{2\phi-1-2d_1})$, noting that $\lambda_j \geq 0$ for all j. We also have $2d_1 + 1 - 2\phi > 0$ when $d - b < 1/2$, it follows that $\Lambda_{p,r}^L(d_1)$ diverges as $T \to \infty$. The asymptotic size of the test follows from the definition of $CV_{\xi,p-r_0}(d, d_1)$. □

Appendix B: Tables for the Empirical Exercise and the Simulation Results

Table 2: Cointegration Test Results for European CDS rates

r_0	One Year CDS Rate JT	p-val(JT)	Λ^L	p-val(Λ^L)	Wavestrapped p-val(Λ^L)	Two Year CDS Rate JT	p-val(JT)	Λ^L	p-val(Λ^L)	Wavestrapped p-val(Λ^L)	Three Year CDS Rate JT	p-val(JT)	Λ^L	p-val(Λ^L)	Wavestrapped p-val(Λ^L)
0	255.988	0.001	15.957	0.000	0.031	235.877	0.001	15.545	0.000	0.097	206.513	0.001	15.245	0.000	0.188
1	184.087	0.001	13.395	0.000	0.105	173.787	0.001	13.144	0.000	0.149	150.881	0.001	12.921	0.000	0.280
2	120.987	0.001	11.040	0.000	0.137	114.865	0.001	10.814	0.001	0.211	102.863	0.001	10.661	0.007	0.302
3	69.024	0.008	8.795	0.002	0.137	66.272	0.014	8.582	0.028	0.289	60.993	0.042	8.447	0.096	0.470
4	40.676	0.045	6.577	0.083	0.527	36.138	0.121	6.427	0.243	0.645	32.206	0.268	6.330	0.419	0.757
5	18.280	0.253	4.627	0.246	0.649	16.758	0.364	4.493	0.499	0.868	16.762	0.364	4.409	0.696	0.961
6	9.033	0.167	2.872	0.360	0.806	8.058	0.252	2.779	0.563	0.556	7.902	0.271	2.718	0.704	0.723
7	1.860	0.208	1.214	0.583	0.724	1.685	0.267	1.187	0.665	0.771	1.749	0.245	1.166	0.754	0.816
	Five Year CDS Rate					Seven Year CDS Rate					Ten Year CDS Rate				
0	196.944	0.001	15.142	0.000	0.270	187.707	0.001	14.959	0.001	0.423	184.439	0.001	14.916	0.000	0.534
1	144.004	0.001	12.696	0.006	0.572	131.046	0.002	12.587	0.019	0.655	123.088	0.008	12.533	0.032	0.756
2	95.707	0.006	10.450	0.060	0.671	85.442	0.039	10.394	0.096	0.781	78.225	0.121	10.378	0.110	0.772
3	54.442	0.137	8.302	0.257	0.691	49.614	0.299	8.277	0.297	0.758	47.256	0.399	8.259	0.338	0.939
4	28.306	0.477	6.257	0.582	0.892	27.710	0.509	6.213	0.662	0.948	25.892	0.606	6.176	0.753	0.979
5	14.187	0.552	4.352	0.805	0.997	14.157	0.554	4.318	0.853	0.997	13.573	0.597	4.296	0.892	1.000
6	7.193	0.353	2.670	0.805	0.806	7.324	0.338	2.634	0.871	0.994	6.922	0.385	2.604	0.922	1.000
7	1.530	0.319	1.146	0.838	0.794	1.451	0.345	1.130	0.905	0.831	1.200	0.429	1.118	0.955	0.844

Note: I use $d_1 = 0.1$ in all wavelet based test statistics. JT stands for Johansen's (1988) trace test and p-val(JT) is the p value of this test. Λ^L is my fractional wavelet variance ratio test (FWVR), where p-val(Λ^L) is the p value of standard FWVR test and wavestrapped p-val(Λ^L) is the wavestrapped p value of the fractional wavelet variance ratio statistic for the same null. I use B=999 wavestrap replication to obtain the wavestrapped distribution of the test statistic. Furthermore, I consider $\delta_t = 0$ which is for the case of no deterministic adjustment. Finally, r_0 indicates the value of the rank under the null hypothesis.

Table 3: Size Distortion Caused by Negative MA Innovation in the Stochastic Trends under the Null Hypothesis $H_0 : r = 0$

T	δ_t	θ	$d_1 = 0.1$: Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	$d_1 = 1$: Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
128	0	-0.9	0.898	0.677	0.176	0.657	0.156	0.658	0.162	0.629	0.077	0.580	0.059	0.451	0.307	0.144	0.291	0.057	0.291	0.064	0.280	0.056	0.226	0.068
		-0.7	0.314	0.159	0.044	0.144	0.039	0.143	0.038	0.128	0.039	0.095	0.042	0.133	0.093	0.047	0.073	0.025	0.086	0.029	0.069	0.044	0.047	0.054
		-0.5	0.125	0.077	0.033	0.070	0.034	0.070	0.033	0.061	0.044	0.042	0.048	0.068	0.056	0.036	0.052	0.032	0.057	0.035	0.049	0.051	0.027	0.045
		0	0.047	0.048	0.042	0.045	0.047	0.045	0.045	0.039	0.053	0.024	0.050	0.047	0.047	0.048	0.040	0.051	0.045	0.048	0.036	0.050	0.021	0.051
		0.5	0.044	0.050	0.050	0.037	0.049	0.043	0.051	0.039	0.057	0.022	0.055	0.047	0.050	0.049	0.040	0.047	0.044	0.046	0.035	0.055	0.021	0.051
		0.7	0.041	0.046	0.051	0.042	0.052	0.043	0.053	0.035	0.052	0.024	0.051	0.046	0.048	0.050	0.044	0.054	0.044	0.044	0.037	0.055	0.021	0.052
		0.9	0.040	0.045	0.050	0.044	0.056	0.045	0.056	0.039	0.054	0.023	0.052	0.046	0.048	0.049	0.044	0.052	0.046	0.051	0.035	0.052	0.021	0.050
	1	-0.9	0.989	0.879	0.342	0.868	0.104	0.883	0.108	0.865	0.044	0.824	0.032	0.689	0.514	0.231	0.541	0.103	0.541	0.110	0.538	0.026	0.536	0.031
		-0.7	0.505	0.243	0.049	0.232	0.029	0.245	0.029	0.219	0.031	0.188	0.033	0.203	0.132	0.038	0.144	0.026	0.143	0.027	0.153	0.030	0.145	0.038
		-0.5	0.176	0.093	0.026	0.089	0.025	0.092	0.025	0.086	0.036	0.058	0.040	0.092	0.072	0.028	0.084	0.030	0.081	0.027	0.088	0.042	0.081	0.045
		0	0.049	0.048	0.037	0.047	0.036	0.048	0.037	0.044	0.053	0.028	0.052	0.051	0.051	0.034	0.055	0.041	0.056	0.038	0.067	0.053	0.060	0.049
		0.5	0.037	0.042	0.040	0.042	0.043	0.044	0.040	0.037	0.057	0.027	0.051	0.044	0.045	0.042	0.054	0.042	0.052	0.043	0.061	0.052	0.061	0.051
		0.7	0.037	0.044	0.044	0.042	0.043	0.044	0.044	0.039	0.056	0.030	0.053	0.046	0.048	0.042	0.052	0.041	0.056	0.044	0.063	0.053	0.061	0.050
		0.9	0.035	0.043	0.043	0.043	0.046	0.046	0.042	0.041	0.056	0.023	0.052	0.048	0.051	0.045	0.050	0.043	0.051	0.042	0.061	0.055	0.059	0.052
	$[1, t]'$	-0.9	0.999	0.949	0.401	0.947	0.174	0.955	0.195	0.939	0.053	0.936	0.031	0.897	0.712	0.367	0.702	0.216	0.718	0.228	0.726	0.035	0.688	0.028
		-0.7	0.708	0.363	0.059	0.352	0.050	0.366	0.048	0.320	0.033	0.318	0.030	0.320	0.191	0.046	0.189	0.059	0.194	0.052	0.211	0.026	0.206	0.035
		-0.5	0.265	0.125	0.024	0.129	0.030	0.133	0.029	0.109	0.038	0.111	0.037	0.129	0.087	0.022	0.086	0.043	0.094	0.041	0.108	0.043	0.108	0.045
		0	0.049	0.047	0.026	0.048	0.037	0.056	0.040	0.037	0.054	0.043	0.050	0.053	0.052	0.029	0.052	0.048	0.057	0.049	0.067	0.054	0.067	0.049
		0.5	0.035	0.044	0.032	0.042	0.047	0.049	0.047	0.035	0.063	0.036	0.053	0.046	0.047	0.033	0.054	0.057	0.051	0.054	0.062	0.056	0.063	0.051
		0.7	0.034	0.045	0.034	0.042	0.048	0.045	0.045	0.033	0.058	0.039	0.052	0.043	0.047	0.028	0.051	0.055	0.055	0.058	0.061	0.055	0.063	0.051
		0.9	0.035	0.046	0.035	0.043	0.045	0.046	0.048	0.034	0.060	0.036	0.052	0.040	0.042	0.029	0.051	0.054	0.053	0.057	0.064	0.058	0.063	0.053
1024	0	-0.9	0.594	0.367	0.046	0.370	0.190	0.361	0.184	0.363	0.086	0.368	0.039	0.184	0.122	0.018	0.124	0.020	0.127	0.020	0.118	0.080	0.119	0.049
		-0.7	0.116	0.077	0.017	0.083	0.047	0.068	0.040	0.071	0.061	0.071	0.044	0.064	0.058	0.029	0.059	0.023	0.059	0.023	0.052	0.054	0.057	0.054
		-0.5	0.066	0.058	0.033	0.056	0.042	0.053	0.043	0.051	0.057	0.055	0.052	0.057	0.055	0.039	0.053	0.032	0.058	0.037	0.055	0.060	0.050	0.050
		0	0.048	0.048	0.041	0.049	0.051	0.051	0.057	0.046	0.059	0.048	0.055	0.048	0.048	0.042	0.053	0.048	0.052	0.047	0.050	0.057	0.044	0.050
		0.5	0.049	0.050	0.043	0.050	0.052	0.043	0.051	0.046	0.060	0.047	0.057	0.052	0.052	0.044	0.054	0.051	0.051	0.050	0.045	0.057	0.046	0.051
		0.7	0.047	0.048	0.047	0.051	0.058	0.043	0.053	0.045	0.056	0.049	0.056	0.050	0.050	0.044	0.053	0.052	0.054	0.051	0.047	0.060	0.047	0.050
		0.9	0.050	0.051	0.047	0.049	0.052	0.044	0.054	0.046	0.066	0.046	0.055	0.052	0.052	0.044	0.055	0.053	0.049	0.049	0.050	0.059	0.047	0.055
	1	-0.9	0.824	0.571	0.072	0.575	0.138	0.581	0.138	0.569	0.037	0.573	0.021	0.279	0.173	0.019	0.170	0.032	0.180	0.030	0.176	0.014	0.183	0.028
		-0.7	0.169	0.098	0.008	0.098	0.011	0.106	0.013	0.094	0.028	0.094	0.038	0.075	0.063	0.019	0.062	0.015	0.062	0.012	0.062	0.036	0.065	0.046
		-0.5	0.071	0.058	0.020	0.064	0.014	0.061	0.015	0.063	0.041	0.056	0.050	0.052	0.049	0.026	0.050	0.018	0.056	0.020	0.053	0.045	0.055	0.052
		0	0.047	0.047	0.035	0.055	0.022	0.048	0.025	0.049	0.052	0.047	0.056	0.049	0.049	0.035	0.047	0.028	0.049	0.028	0.052	0.051	0.054	0.054
		0.5	0.043	0.044	0.035	0.048	0.029	0.051	0.027	0.050	0.052	0.047	0.057	0.048	0.049	0.039	0.049	0.034	0.056	0.036	0.051	0.052	0.052	0.053
		0.7	0.042	0.043	0.035	0.045	0.027	0.045	0.026	0.046	0.054	0.044	0.058	0.050	0.050	0.037	0.043	0.030	0.054	0.034	0.050	0.053	0.053	0.053
		0.9	0.044	0.046	0.036	0.049	0.028	0.045	0.025	0.047	0.053	0.046	0.057	0.050	0.051	0.042	0.047	0.032	0.049	0.030	0.049	0.052	0.052	0.052
	$[1, t]'$	-0.9	0.975	0.817	0.140	0.813	0.300	0.815	0.291	0.807	0.057	0.802	0.023	0.453	0.282	0.029	0.299	0.098	0.277	0.092	0.290	0.021	0.293	0.024
		-0.7	0.281	0.134	0.005	0.136	0.024	0.133	0.022	0.127	0.026	0.124	0.036	0.086	0.065	0.011	0.068	0.025	0.068	0.025	0.071	0.033	0.078	0.046
		-0.5	0.099	0.071	0.011	0.068	0.016	0.067	0.018	0.063	0.040	0.063	0.050	0.057	0.051	0.021	0.057	0.029	0.051	0.027	0.060	0.048	0.063	0.051
		0	0.055	0.056	0.024	0.050	0.026	0.049	0.029	0.043	0.053	0.050	0.059	0.046	0.046	0.025	0.051	0.035	0.049	0.036	0.054	0.053	0.053	0.055
		0.5	0.045	0.048	0.024	0.047	0.028	0.045	0.031	0.042	0.054	0.046	0.060	0.042	0.043	0.026	0.051	0.036	0.050	0.037	0.051	0.051	0.051	0.057
		0.7	0.050	0.053	0.031	0.053	0.034	0.049	0.031	0.046	0.055	0.044	0.059	0.046	0.047	0.027	0.054	0.037	0.052	0.042	0.051	0.052	0.049	0.054
		0.9	0.045	0.047	0.025	0.052	0.030	0.044	0.032	0.041	0.057	0.041	0.061	0.048	0.049	0.032	0.051	0.039	0.045	0.036	0.053	0.052	0.055	0.055

Note: The simulations are based on the DGP (a). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 0$ and $p = 2$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 0$ and $p = 2$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, θ demonstrates the MA(1) coefficient of the innovation process $\epsilon_{2,t}$.

Table 4: Size Distortion Caused by Negative MA Innovation in the Stochastic Trends under the Null Hypothesis $H_0 : r = 1$

			$d_1 = 0.1$											$d_1 = 1$										
T	δ_t	θ	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
		-0.9	0.892	0.660	0.130	0.630	0.118	0.622	0.126	0.607	0.048	0.545	0.045	0.434	0.264	0.085	0.225	0.025	0.220	0.021	0.205	0.022	0.163	0.029
		-0.7	0.319	0.153	0.032	0.134	0.037	0.139	0.035	0.128	0.037	0.093	0.033	0.122	0.073	0.026	0.062	0.015	0.061	0.014	0.048	0.028	0.035	0.035
		-0.5	0.120	0.069	0.025	0.067	0.031	0.068	0.032	0.055	0.043	0.040	0.043	0.064	0.047	0.024	0.039	0.021	0.038	0.020	0.034	0.035	0.018	0.037
	0	0	0.048	0.047	0.039	0.042	0.042	0.041	0.039	0.036	0.047	0.024	0.047	0.046	0.039	0.029	0.028	0.028	0.030	0.033	0.027	0.043	0.014	0.036
		0.5	0.042	0.045	0.045	0.041	0.047	0.041	0.047	0.033	0.045	0.025	0.045	0.046	0.040	0.036	0.028	0.031	0.031	0.035	0.027	0.044	0.014	0.037
		0.7	0.042	0.043	0.040	0.045	0.051	0.039	0.041	0.036	0.046	0.023	0.045	0.042	0.038	0.033	0.033	0.035	0.031	0.032	0.026	0.039	0.016	0.037
		0.9	0.041	0.044	0.037	0.039	0.040	0.039	0.047	0.035	0.049	0.021	0.047	0.048	0.042	0.035	0.031	0.038	0.030	0.033	0.023	0.035	0.013	0.038
		-0.9	0.988	0.865	0.238	0.865	0.069	0.860	0.065	0.836	0.028	0.806	0.020	0.674	0.459	0.144	0.467	0.046	0.454	0.043	0.469	0.012	0.483	0.015
		-0.7	0.492	0.227	0.037	0.240	0.023	0.229	0.022	0.190	0.024	0.182	0.025	0.188	0.108	0.027	0.118	0.016	0.114	0.018	0.121	0.022	0.126	0.031
		-0.5	0.176	0.092	0.020	0.090	0.019	0.083	0.020	0.072	0.029	0.061	0.033	0.083	0.059	0.021	0.063	0.017	0.056	0.017	0.073	0.031	0.075	0.037
128	1	0	0.049	0.048	0.028	0.049	0.028	0.042	0.029	0.035	0.043	0.032	0.042	0.043	0.036	0.023	0.047	0.028	0.043	0.029	0.050	0.039	0.054	0.039
		0.5	0.039	0.040	0.033	0.046	0.034	0.037	0.032	0.029	0.045	0.025	0.042	0.041	0.039	0.029	0.046	0.031	0.040	0.031	0.050	0.041	0.049	0.039
		0.7	0.034	0.038	0.030	0.044	0.034	0.039	0.033	0.033	0.044	0.025	0.041	0.041	0.037	0.029	0.044	0.030	0.045	0.030	0.048	0.042	0.053	0.039
		0.9	0.033	0.037	0.029	0.046	0.035	0.043	0.034	0.033	0.044	0.025	0.041	0.040	0.037	0.029	0.045	0.034	0.042	0.028	0.048	0.043	0.054	0.039
		-0.9	0.999	0.936	0.275	0.931	0.107	0.928	0.103	0.914	0.029	0.914	0.015	0.863	0.630	0.224	0.635	0.102	0.637	0.114	0.639	0.010	0.576	0.010
		-0.7	0.688	0.326	0.039	0.327	0.031	0.325	0.029	0.296	0.021	0.301	0.022	0.285	0.151	0.025	0.173	0.033	0.167	0.037	0.181	0.016	0.169	0.021
		-0.5	0.268	0.120	0.016	0.114	0.022	0.120	0.020	0.102	0.025	0.101	0.028	0.112	0.066	0.016	0.081	0.030	0.072	0.024	0.086	0.028	0.086	0.031
	$[1, t]'$	0	0.050	0.047	0.019	0.044	0.027	0.045	0.030	0.040	0.040	0.040	0.035	0.039	0.033	0.015	0.046	0.035	0.042	0.036	0.051	0.038	0.052	0.034
		0.5	0.034	0.042	0.023	0.044	0.031	0.038	0.031	0.033	0.043	0.036	0.039	0.039	0.037	0.022	0.044	0.040	0.042	0.038	0.047	0.042	0.051	0.034
		0.7	0.030	0.038	0.021	0.037	0.031	0.036	0.031	0.031	0.042	0.035	0.039	0.036	0.035	0.021	0.043	0.039	0.043	0.036	0.048	0.039	0.047	0.035
		0.9	0.030	0.036	0.022	0.039	0.034	0.035	0.030	0.034	0.043	0.031	0.038	0.036	0.035	0.021	0.043	0.037	0.042	0.040	0.047	0.038	0.046	0.033
		-0.9	0.601	0.372	0.048	0.371	0.188	0.370	0.187	0.358	0.097	0.372	0.036	0.180	0.119	0.019	0.118	0.018	0.114	0.015	0.119	0.072	0.109	0.047
		-0.7	0.111	0.073	0.015	0.081	0.046	0.081	0.045	0.070	0.057	0.077	0.044	0.066	0.059	0.028	0.051	0.019	0.047	0.017	0.053	0.057	0.051	0.045
		-0.5	0.065	0.055	0.028	0.057	0.042	0.055	0.037	0.054	0.057	0.053	0.051	0.056	0.054	0.034	0.051	0.032	0.047	0.033	0.048	0.059	0.046	0.049
	0	0	0.050	0.050	0.037	0.049	0.046	0.048	0.047	0.045	0.063	0.049	0.053	0.058	0.058	0.053	0.046	0.042	0.046	0.043	0.047	0.058	0.047	0.051
		0.5	0.047	0.048	0.042	0.051	0.048	0.053	0.054	0.043	0.062	0.051	0.056	0.056	0.056	0.050	0.045	0.045	0.045	0.045	0.050	0.058	0.047	0.049
		0.7	0.048	0.050	0.042	0.052	0.054	0.052	0.051	0.047	0.063	0.054	0.051	0.051	0.050	0.039	0.047	0.042	0.047	0.050	0.051	0.056	0.047	0.051
		0.9	0.045	0.046	0.040	0.050	0.053	0.053	0.050	0.042	0.056	0.048	0.057	0.053	0.053	0.045	0.047	0.045	0.044	0.044	0.051	0.054	0.043	0.049
		-0.9	0.830	0.574	0.066	0.590	0.129	0.580	0.133	0.586	0.038	0.560	0.022	0.282	0.179	0.015	0.176	0.029	0.174	0.029	0.176	0.014	0.181	0.029
		-0.7	0.178	0.100	0.008	0.093	0.011	0.096	0.012	0.102	0.025	0.092	0.037	0.076	0.066	0.019	0.064	0.011	0.065	0.012	0.055	0.036	0.068	0.045
		-0.5	0.073	0.057	0.019	0.065	0.014	0.057	0.014	0.057	0.039	0.056	0.048	0.059	0.056	0.032	0.051	0.018	0.051	0.017	0.056	0.045	0.055	0.049
1024	1	0	0.049	0.050	0.031	0.047	0.023	0.048	0.022	0.051	0.049	0.044	0.055	0.053	0.053	0.035	0.053	0.026	0.051	0.029	0.050	0.048	0.055	0.051
		0.5	0.048	0.049	0.035	0.049	0.022	0.046	0.024	0.043	0.050	0.041	0.056	0.047	0.046	0.034	0.055	0.030	0.046	0.028	0.048	0.050	0.059	0.052
		0.7	0.046	0.047	0.037	0.045	0.023	0.042	0.022	0.047	0.049	0.043	0.055	0.055	0.055	0.039	0.054	0.034	0.048	0.028	0.047	0.048	0.053	0.051
		0.9	0.051	0.053	0.036	0.051	0.025	0.045	0.024	0.051	0.050	0.044	0.055	0.051	0.051	0.039	0.053	0.029	0.050	0.030	0.051	0.050	0.059	0.051
		-0.9	0.972	0.812	0.123	0.814	0.276	0.808	0.294	0.806	0.056	0.807	0.021	0.458	0.294	0.028	0.273	0.091	0.289	0.096	0.274	0.021	0.299	0.026
		-0.7	0.268	0.136	0.004	0.137	0.022	0.132	0.024	0.121	0.025	0.137	0.034	0.089	0.066	0.010	0.068	0.020	0.071	0.021	0.068	0.034	0.081	0.043
		-0.5	0.093	0.066	0.012	0.070	0.016	0.067	0.017	0.059	0.036	0.063	0.047	0.064	0.057	0.017	0.051	0.024	0.055	0.029	0.051	0.045	0.059	0.050
	$[1, t]'$	0	0.052	0.051	0.021	0.051	0.026	0.051	0.025	0.043	0.047	0.047	0.056	0.053	0.053	0.026	0.047	0.031	0.051	0.030	0.050	0.051	0.056	0.052
		0.5	0.043	0.044	0.024	0.050	0.025	0.043	0.026	0.042	0.049	0.047	0.057	0.051	0.052	0.027	0.051	0.037	0.049	0.036	0.044	0.050	0.050	0.053
		0.7	0.047	0.049	0.025	0.044	0.027	0.047	0.027	0.038	0.050	0.046	0.058	0.048	0.047	0.026	0.049	0.036	0.049	0.031	0.050	0.050	0.052	0.055
		0.9	0.046	0.047	0.023	0.043	0.027	0.047	0.027	0.045	0.052	0.046	0.057	0.052	0.052	0.024	0.047	0.035	0.047	0.032	0.045	0.051	0.058	0.053

Note: The simulations are based on the DGP (b). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, θ demonstrates the MA(1) coefficient of the innovation process $\epsilon_{2,t}$.

Table 5: Size Distortion Caused by Negative MA Innovations of the Cointegration Residuals under the Null Hypothesis $H_0 : r = 1$

			$d_1 = 0.1$											$d_1 = 1$										
T	δ_t	θ	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
		-0.9	0.046	0.046	0.031	0.047	0.040	0.045	0.039	0.035	0.047	0.026	0.044	0.052	0.052	0.037	0.043	0.039	0.045	0.037	0.032	0.048	0.020	0.050
		-0.7	0.049	0.049	0.033	0.043	0.041	0.045	0.040	0.035	0.050	0.025	0.045	0.052	0.051	0.034	0.043	0.037	0.045	0.038	0.035	0.049	0.018	0.047
		-0.5	0.046	0.047	0.035	0.037	0.035	0.044	0.037	0.036	0.048	0.023	0.048	0.055	0.052	0.038	0.041	0.035	0.042	0.035	0.028	0.043	0.016	0.043
	0	0	0.049	0.048	0.034	0.041	0.040	0.044	0.041	0.037	0.046	0.023	0.045	0.048	0.041	0.031	0.032	0.031	0.033	0.030	0.026	0.036	0.011	0.039
		0.5	0.051	0.050	0.036	0.040	0.041	0.042	0.039	0.033	0.045	0.024	0.044	0.040	0.036	0.028	0.030	0.026	0.032	0.028	0.021	0.034	0.011	0.038
		0.7	0.048	0.047	0.034	0.044	0.040	0.044	0.041	0.032	0.044	0.022	0.043	0.044	0.039	0.028	0.031	0.029	0.030	0.030	0.022	0.037	0.011	0.034
		0.9	0.048	0.047	0.035	0.042	0.039	0.049	0.042	0.034	0.042	0.023	0.041	0.040	0.036	0.025	0.030	0.029	0.033	0.027	0.025	0.035	0.011	0.031
		-0.9	0.049	0.050	0.021	0.047	0.024	0.047	0.024	0.033	0.040	0.036	0.040	0.052	0.053	0.027	0.050	0.031	0.058	0.030	0.060	0.046	0.066	0.046
		-0.7	0.051	0.052	0.021	0.047	0.025	0.049	0.025	0.035	0.042	0.029	0.041	0.049	0.049	0.027	0.049	0.030	0.057	0.033	0.061	0.046	0.065	0.044
		-0.5	0.050	0.051	0.024	0.046	0.026	0.041	0.026	0.030	0.040	0.035	0.041	0.055	0.054	0.033	0.047	0.032	0.054	0.030	0.064	0.044	0.063	0.043
128	1	0	0.052	0.053	0.028	0.044	0.027	0.049	0.029	0.031	0.041	0.030	0.042	0.048	0.045	0.027	0.042	0.032	0.047	0.029	0.053	0.039	0.055	0.040
		0.5	0.047	0.049	0.026	0.046	0.028	0.044	0.028	0.035	0.041	0.032	0.039	0.042	0.040	0.024	0.037	0.026	0.046	0.029	0.053	0.039	0.054	0.038
		0.7	0.047	0.049	0.026	0.045	0.026	0.041	0.027	0.032	0.039	0.030	0.036	0.047	0.045	0.026	0.042	0.026	0.044	0.027	0.043	0.036	0.053	0.039
		0.9	0.051	0.050	0.026	0.042	0.025	0.041	0.024	0.033	0.036	0.032	0.034	0.042	0.040	0.022	0.041	0.025	0.048	0.025	0.051	0.037	0.055	0.036
		-0.9	0.049	0.048	0.015	0.046	0.022	0.054	0.022	0.043	0.034	0.046	0.035	0.054	0.052	0.019	0.053	0.036	0.049	0.042	0.061	0.042	0.063	0.041
		-0.7	0.055	0.055	0.016	0.045	0.023	0.052	0.025	0.042	0.035	0.042	0.036	0.053	0.051	0.020	0.055	0.043	0.051	0.041	0.065	0.041	0.060	0.042
		-0.5	0.048	0.048	0.016	0.050	0.028	0.049	0.026	0.044	0.039	0.039	0.038	0.054	0.052	0.021	0.050	0.033	0.045	0.040	0.060	0.041	0.056	0.041
	$[1, t]'$	0	0.048	0.047	0.018	0.046	0.025	0.048	0.026	0.042	0.041	0.040	0.036	0.055	0.048	0.020	0.046	0.038	0.041	0.036	0.053	0.036	0.048	0.034
		0.5	0.047	0.044	0.015	0.042	0.026	0.043	0.027	0.040	0.035	0.040	0.035	0.046	0.041	0.016	0.044	0.035	0.037	0.032	0.053	0.035	0.045	0.032
		0.7	0.048	0.047	0.016	0.041	0.023	0.048	0.027	0.042	0.034	0.041	0.031	0.043	0.038	0.015	0.045	0.034	0.040	0.034	0.052	0.035	0.049	0.030
		0.9	0.046	0.046	0.016	0.048	0.027	0.046	0.023	0.041	0.032	0.041	0.030	0.048	0.044	0.015	0.045	0.033	0.038	0.031	0.052	0.032	0.047	0.032
		-0.9	0.052	0.053	0.034	0.050	0.050	0.044	0.048	0.048	0.059	0.043	0.053	0.050	0.050	0.044	0.052	0.050	0.051	0.040	0.050	0.055	0.045	0.056
		-0.7	0.050	0.051	0.039	0.050	0.047	0.043	0.043	0.046	0.057	0.047	0.054	0.049	0.049	0.039	0.049	0.043	0.050	0.045	0.047	0.057	0.048	0.055
		-0.5	0.049	0.049	0.038	0.048	0.045	0.045	0.048	0.049	0.058	0.043	0.057	0.052	0.052	0.048	0.045	0.039	0.047	0.044	0.050	0.055	0.046	0.054
	0	0	0.051	0.051	0.040	0.051	0.054	0.046	0.048	0.044	0.061	0.044	0.054	0.051	0.050	0.039	0.049	0.046	0.047	0.044	0.049	0.057	0.045	0.052
		0.5	0.056	0.056	0.040	0.047	0.046	0.045	0.046	0.046	0.057	0.045	0.049	0.050	0.049	0.042	0.043	0.041	0.048	0.039	0.049	0.056	0.044	0.049
		0.7	0.053	0.054	0.040	0.049	0.047	0.045	0.048	0.049	0.057	0.051	0.053	0.051	0.050	0.043	0.045	0.043	0.050	0.043	0.049	0.058	0.047	0.052
		0.9	0.051	0.051	0.035	0.048	0.050	0.045	0.047	0.045	0.055	0.044	0.053	0.048	0.048	0.038	0.046	0.044	0.047	0.042	0.043	0.055	0.045	0.052
		-0.9	0.048	0.047	0.026	0.048	0.021	0.047	0.019	0.053	0.047	0.045	0.055	0.050	0.051	0.036	0.046	0.027	0.045	0.025	0.052	0.050	0.052	0.051
		-0.7	0.048	0.048	0.028	0.045	0.019	0.048	0.022	0.055	0.049	0.048	0.054	0.051	0.051	0.040	0.046	0.025	0.049	0.025	0.056	0.050	0.054	0.052
		-0.5	0.047	0.047	0.029	0.043	0.018	0.052	0.021	0.053	0.052	0.051	0.055	0.051	0.051	0.031	0.046	0.025	0.055	0.029	0.054	0.051	0.056	0.052
1024	1	0.000	0.047	0.047	0.032	0.046	0.023	0.051	0.021	0.047	0.049	0.046	0.054	0.051	0.051	0.034	0.046	0.026	0.047	0.024	0.053	0.049	0.054	0.050
		0.5	0.048	0.048	0.035	0.042	0.022	0.044	0.021	0.050	0.047	0.043	0.052	0.053	0.052	0.036	0.046	0.027	0.049	0.028	0.053	0.050	0.054	0.052
		0.7	0.045	0.044	0.027	0.042	0.019	0.050	0.021	0.051	0.047	0.046	0.051	0.050	0.049	0.033	0.047	0.025	0.048	0.028	0.051	0.048	0.054	0.050
		0.9	0.051	0.050	0.033	0.041	0.020	0.050	0.020	0.049	0.044	0.049	0.051	0.048	0.048	0.031	0.047	0.025	0.049	0.025	0.049	0.049	0.053	0.050
		-0.9	0.049	0.049	0.019	0.052	0.024	0.047	0.023	0.048	0.048	0.050	0.058	0.051	0.051	0.023	0.045	0.034	0.049	0.033	0.052	0.051	0.057	0.054
		-0.7	0.055	0.055	0.020	0.054	0.022	0.045	0.025	0.048	0.048	0.049	0.055	0.051	0.051	0.027	0.047	0.033	0.050	0.034	0.054	0.049	0.055	0.053
		-0.5	0.047	0.047	0.018	0.056	0.023	0.043	0.021	0.053	0.048	0.048	0.055	0.053	0.052	0.025	0.048	0.030	0.049	0.033	0.051	0.051	0.052	0.054
	$[1, t]'$	0	0.050	0.050	0.022	0.052	0.024	0.052	0.026	0.053	0.049	0.049	0.056	0.055	0.055	0.030	0.046	0.031	0.051	0.036	0.054	0.052	0.058	0.053
		0.5	0.051	0.051	0.020	0.052	0.024	0.048	0.025	0.049	0.046	0.049	0.054	0.053	0.052	0.024	0.049	0.036	0.049	0.033	0.049	0.048	0.056	0.052
		0.7	0.049	0.050	0.017	0.054	0.024	0.049	0.022	0.051	0.044	0.052	0.053	0.052	0.052	0.021	0.046	0.031	0.048	0.032	0.055	0.050	0.055	0.051
		0.9	0.051	0.051	0.019	0.051	0.020	0.048	0.022	0.049	0.044	0.051	0.051	0.053	0.053	0.023	0.043	0.033	0.053	0.033	0.052	0.049	0.055	0.052

Note: The simulations are based on DGP (c). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, θ demonstrates the MA(1) coefficient of the cointegrating residuals $u_{2,t}$.

Table 6: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 0$ with Sample Size=128

				$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}
T	θ	δ_t	ρ_c		Haar		D_2		S_2		S_4		S_8			Haar		D_2		S_2		S_4		S_8	
128	-0.9	0	0.1	0.365	0.400	0.849	1.000	0.598	1.000	0.592	1.000	0.267	0.999	0.207	0.969	0.887	0.670	0.872	0.358	0.874	0.362	0.852	0.168	0.790	0.177
			0.2	0.519	0.596	0.992	1.000	0.873	1.000	0.902	1.000	0.343	1.000	0.288	0.999	0.990	0.938	0.981	0.681	0.983	0.683	0.978	0.335	0.956	0.321
			0.3	0.663	0.775	1.000	1.000	0.970	1.000	0.974	1.000	0.414	1.000	0.308	1.000	0.999	0.985	0.996	0.839	0.998	0.864	0.996	0.549	0.991	0.473
			0.4	0.788	0.885	1.000	1.000	0.991	1.000	0.997	1.000	0.444	1.000	0.336	1.000	1.000	0.998	0.999	0.924	1.000	0.919	0.999	0.678	0.997	0.557
			0.5	0.891	0.949	1.000	1.000	0.998	1.000	0.995	1.000	0.476	1.000	0.375	1.000	1.000	0.999	1.000	0.955	1.000	0.958	1.000	0.764	1.000	0.670
		1	0.1	0.220	0.248	0.829	1.000	0.231	1.000	0.229	1.000	0.067	1.000	0.037	0.996	0.964	0.744	0.962	0.349	0.960	0.356	0.961	0.042	0.953	0.025
			0.2	0.361	0.422	0.977	1.000	0.341	1.000	0.374	1.000	0.085	1.000	0.042	1.000	0.999	0.970	0.997	0.706	0.998	0.707	0.996	0.079	0.994	0.024
			0.3	0.482	0.552	0.999	1.000	0.454	1.000	0.456	1.000	0.109	1.000	0.049	1.000	1.000	0.998	1.000	0.869	1.000	0.869	1.000	0.141	0.999	0.024
			0.4	0.584	0.657	1.000	1.000	0.506	1.000	0.496	1.000	0.129	1.000	0.052	1.000	1.000	0.999	1.000	0.920	1.000	0.920	1.000	0.233	1.000	0.030
			0.5	0.656	0.718	1.000	1.000	0.554	1.000	0.541	1.000	0.136	1.000	0.057	1.000	1.000	1.000	1.000	0.944	1.000	0.948	1.000	0.304	1.000	0.046
		$[1, t]'$	0.1	0.141	0.153	0.706	0.999	0.305	0.999	0.301	0.999	0.073	0.999	0.038	0.999	0.977	0.788	0.973	0.477	0.969	0.468	0.967	0.051	0.931	0.025
			0.2	0.250	0.294	0.890	1.000	0.443	1.000	0.448	1.000	0.095	1.000	0.043	1.000	1.000	0.981	0.997	0.750	0.996	0.748	0.994	0.087	0.985	0.022
			0.3	0.349	0.408	0.964	1.000	0.554	1.000	0.543	1.000	0.118	1.000	0.048	1.000	1.000	0.999	0.999	0.846	0.999	0.857	0.998	0.155	0.997	0.025
			0.4	0.434	0.494	0.991	1.000	0.616	1.000	0.639	1.000	0.148	1.000	0.056	1.000	1.000	1.000	1.000	0.885	0.999	0.896	0.999	0.217	1.000	0.032
			0.5	0.520	0.570	0.997	1.000	0.669	1.000	0.692	1.000	0.164	1.000	0.059	1.000	1.000	1.000	1.000	0.915	1.000	0.913	1.000	0.283	1.000	0.040
	-0.5	0	0.1	0.517	0.410	0.334	0.495	0.111	0.498	0.120	0.454	0.075	0.384	0.079	0.331	0.249	0.090	0.230	0.046	0.246	0.045	0.217	0.051	0.172	0.064
			0.2	0.938	0.808	0.725	0.876	0.247	0.869	0.230	0.862	0.117	0.778	0.115	0.654	0.523	0.248	0.505	0.091	0.517	0.095	0.485	0.081	0.407	0.081
			0.3	0.999	0.972	0.947	0.985	0.391	0.984	0.369	0.981	0.173	0.960	0.160	0.841	0.713	0.408	0.685	0.167	0.701	0.168	0.673	0.095	0.583	0.121
			0.4	1.000	0.997	0.993	0.999	0.531	0.998	0.504	0.998	0.232	0.995	0.194	0.926	0.826	0.567	0.795	0.259	0.801	0.262	0.781	0.136	0.712	0.134
			0.5	1.000	1.000	1.000	1.000	0.661	1.000	0.633	1.000	0.267	1.000	0.217	0.968	0.895	0.686	0.870	0.346	0.870	0.341	0.858	0.157	0.790	0.174
		1	0.1	0.427	0.346	0.246	0.510	0.039	0.514	0.038	0.487	0.036	0.450	0.038	0.421	0.314	0.074	0.334	0.040	0.335	0.039	0.344	0.033	0.367	0.040
			0.2	0.874	0.738	0.614	0.869	0.063	0.876	0.070	0.859	0.041	0.826	0.034	0.786	0.644	0.239	0.653	0.097	0.656	0.098	0.658	0.032	0.681	0.034
			0.3	0.993	0.942	0.874	0.985	0.116	0.984	0.119	0.980	0.049	0.972	0.035	0.941	0.843	0.456	0.844	0.167	0.845	0.168	0.846	0.033	0.846	0.031
			0.4	1.000	0.993	0.975	0.999	0.175	0.999	0.177	0.998	0.063	0.997	0.036	0.983	0.928	0.618	0.927	0.261	0.927	0.266	0.919	0.037	0.919	0.026
			0.5	1.000	0.999	0.996	1.000	0.236	1.000	0.242	1.000	0.062	0.999	0.037	0.996	0.964	0.747	0.958	0.373	0.962	0.363	0.960	0.039	0.956	0.026
		$[1, t]'$	0.1	0.271	0.237	0.130	0.422	0.045	0.430	0.048	0.409	0.037	0.433	0.036	0.404	0.277	0.060	0.316	0.073	0.299	0.072	0.316	0.034	0.292	0.037
			0.2	0.687	0.572	0.389	0.788	0.093	0.791	0.098	0.770	0.039	0.784	0.033	0.789	0.612	0.209	0.649	0.151	0.623	0.153	0.641	0.035	0.586	0.032
			0.3	0.936	0.827	0.658	0.950	0.153	0.951	0.153	0.940	0.047	0.953	0.036	0.950	0.833	0.420	0.848	0.250	0.837	0.270	0.839	0.037	0.771	0.029
			0.4	0.994	0.946	0.846	0.993	0.237	0.992	0.237	0.989	0.063	0.991	0.035	0.992	0.936	0.618	0.938	0.387	0.934	0.366	0.929	0.042	0.878	0.031
			0.5	1.000	0.989	0.947	0.999	0.298	0.999	0.303	0.999	0.075	0.999	0.037	0.999	0.979	0.766	0.973	0.488	0.969	0.475	0.966	0.048	0.936	0.025
	0	0	0.1	0.335	0.318	0.297	0.308	0.101	0.298	0.106	0.289	0.072	0.226	0.058	0.168	0.162	0.082	0.160	0.061	0.176	0.059	0.151	0.056	0.108	0.056
			0.2	0.705	0.630	0.603	0.615	0.183	0.604	0.180	0.593	0.100	0.494	0.078	0.393	0.369	0.178	0.359	0.102	0.365	0.089	0.335	0.074	0.266	0.063
			0.3	0.925	0.845	0.826	0.841	0.257	0.833	0.268	0.820	0.124	0.736	0.104	0.587	0.538	0.292	0.506	0.130	0.534	0.140	0.489	0.084	0.410	0.087
			0.4	0.990	0.951	0.942	0.945	0.334	0.948	0.316	0.939	0.144	0.883	0.133	0.700	0.635	0.359	0.619	0.176	0.637	0.173	0.610	0.103	0.517	0.097
			0.5	0.999	0.986	0.982	0.984	0.397	0.983	0.374	0.980	0.177	0.955	0.146	0.794	0.721	0.444	0.693	0.217	0.707	0.216	0.686	0.117	0.596	0.112
		1	0.1	0.277	0.257	0.217	0.280	0.044	0.278	0.041	0.263	0.054	0.229	0.050	0.214	0.209	0.066	0.226	0.047	0.220	0.048	0.224	0.052	0.247	0.050
			0.2	0.626	0.551	0.491	0.577	0.054	0.581	0.057	0.563	0.058	0.516	0.050	0.477	0.442	0.159	0.467	0.070	0.475	0.077	0.477	0.051	0.502	0.049
			0.3	0.883	0.793	0.743	0.809	0.076	0.809	0.082	0.799	0.056	0.761	0.050	0.695	0.638	0.273	0.669	0.116	0.663	0.104	0.666	0.048	0.675	0.047
			0.4	0.982	0.924	0.892	0.928	0.102	0.930	0.107	0.925	0.061	0.902	0.048	0.821	0.758	0.377	0.777	0.163	0.779	0.170	0.789	0.051	0.787	0.049
			0.5	0.997	0.973	0.961	0.977	0.131	0.979	0.125	0.975	0.061	0.967	0.046	0.910	0.849	0.482	0.847	0.200	0.852	0.213	0.854	0.049	0.851	0.044
		$[1, t]'$	0.1	0.185	0.176	0.107	0.184	0.043	0.196	0.048	0.187	0.056	0.191	0.049	0.156	0.148	0.049	0.186	0.074	0.170	0.075	0.198	0.049	0.173	0.051
			0.2	0.474	0.418	0.292	0.451	0.075	0.450	0.069	0.435	0.055	0.457	0.048	0.390	0.354	0.114	0.426	0.117	0.407	0.125	0.432	0.056	0.386	0.051
			0.3	0.759	0.656	0.514	0.679	0.106	0.693	0.102	0.670	0.059	0.695	0.048	0.639	0.571	0.208	0.635	0.193	0.621	0.189	0.638	0.053	0.561	0.050
			0.4	0.926	0.819	0.701	0.842	0.134	0.843	0.130	0.839	0.066	0.850	0.048	0.803	0.718	0.315	0.769	0.249	0.760	0.241	0.768	0.053	0.698	0.045
			0.5	0.985	0.919	0.835	0.930	0.169	0.927	0.165	0.930	0.069	0.938	0.043	0.903	0.827	0.437	0.849	0.318	0.847	0.319	0.848	0.056	0.769	0.046
	0.5	0	0.1	0.310	0.309	0.298	0.270	0.111	0.271	0.109	0.256	0.073	0.197	0.061	0.155	0.162	0.092	0.153	0.064	0.164	0.064	0.136	0.055	0.106	0.056
			0.2	0.626	0.604	0.594	0.555	0.187	0.553	0.172	0.541	0.095	0.449	0.073	0.351	0.351	0.185	0.340	0.105	0.357	0.104	0.311	0.069	0.257	0.069
			0.3	0.860	0.820	0.810	0.786	0.255	0.785	0.259	0.768	0.116	0.677	0.094	0.519	0.506	0.281	0.486	0.154	0.490	0.149	0.472	0.088	0.389	0.076
			0.4	0.963	0.932	0.927	0.909	0.294	0.908	0.320	0.902	0.142	0.827	0.117	0.627	0.603	0.343	0.584	0.182	0.605	0.187	0.576	0.101	0.476	0.093
			0.5	0.993	0.976	0.974	0.963	0.370	0.964	0.359	0.960	0.164	0.917	0.129	0.709	0.673	0.396	0.657	0.226	0.685	0.201	0.648	0.110	0.561	0.103
		1	0.1	0.248	0.252	0.220	0.247	0.047	0.250	0.051	0.227	0.060	0.201	0.053	0.185	0.192	0.074	0.211	0.054	0.210	0.057	0.214	0.057	0.231	0.053
			0.2	0.554	0.541	0.494	0.513	0.064	0.518	0.067	0.507	0.062	0.469	0.055	0.414	0.415	0.152	0.447	0.091	0.439	0.082	0.456	0.060	0.474	0.055
			0.3	0.803	0.769	0.725	0.752	0.090	0.754	0.089	0.733	0.067	0.700	0.055	0.613	0.599	0.259	0.634	0.128	0.623	0.122	0.624	0.057	0.647	0.053
			0.4	0.935	0.899	0.870	0.885	0.112	0.884	0.104	0.883	0.068	0.848	0.056	0.737	0.715	0.365	0.752	0.162	0.745	0.162	0.751	0.058	0.755	0.055
			0.5	0.982	0.958	0.942	0.953	0.131	0.958	0.129	0.951	0.075	0.935	0.057	0.830	0.797	0.437	0.818	0.195	0.820	0.201	0.822	0.061	0.820	0.051
		$[1, t]'$	0.1	0.170	0.176	0.122	0.167	0.051	0.163	0.055	0.157	0.062	0.168	0.054	0.126	0.133	0.053	0.166	0.071	0.160	0.080	0.187	0.059	0.162	0.052
			0.2	0.418	0.414	0.319	0.390	0.075	0.394	0.076	0.372	0.063	0.399	0.056	0.319	0.324	0.121	0.390	0.139	0.369	0.126	0.408	0.065	0.355	0.054
			0.3	0.657	0.636	0.524	0.604	0.100	0.610	0.107	0.595	0.074	0.627	0.057	0.528	0.517	0.212	0.601	0.192	0.587	0.193	0.609	0.062	0.521	0.056
			0.4	0.823	0.781	0.685	0.774	0.141	0.777	0.137	0.768	0.080	0.794	0.054	0.697	0.669	0.326	0.734	0.254	0.720	0.251	0.746	0.065	0.646	0.053
			0.5	0.937	0.886	0.812	0.874	0.158	0.879	0.164	0.867	0.083	0.889	0.056	0.805	0.769	0.405	0.818	0.332	0.811	0.317	0.822	0.065	0.733	0.053
	0.9	0	0.1	0.298	0.297	0.299	0.273	0.098	0.273	0.109	0.258	0.065	0.203	0.062	0.154	0.160	0.091	0.150	0.066	0.168	0.070	0.140	0.057	0.098	0.055
			0.2	0.604	0.587	0.591	0.556	0.182	0.546	0.195	0.523	0.095	0.443	0.073	0.340	0.342	0.176	0.334	0.107	0.361	0.116	0.321	0.070	0.245	0.063
			0.3	0.837	0.805	0.808	0.779	0.247	0.781	0.261	0.763	0.117	0.667	0.096	0.512	0.503	0.270	0.483	0.145	0.498	0.155	0.465	0.084	0.386	0.078
			0.4	0.945	0.916	0.918	0.905	0.333	0.902	0.319	0.898	0.142	0.826	0.113	0.619	0.600	0.332	0.592	0.192	0.601	0.188	0.568	0.091	0.487	0.086
			0.5	0.986	0.967	0.968	0.963	0.379	0.957	0.385	0.957	0.158	0.913	0.128	0.700	0.674	0.403	0.664	0.226	0.673	0.219	0.645	0.108	0.566	0.102
		1	0.1	0.255	0.263	0.237	0.244	0.050	0.241	0.048	0.230	0.059	0.194	0.054	0.184	0.194	0.080	0.206	0.053	0.208	0.059	0.218	0.056	0.229	0.052
			0.2	0.541	0.537	0.499	0.513	0.064	0.509	0.066	0.496	0.059	0.453	0.056	0.405	0.411	0.162	0.444	0.082	0.451	0.081	0.457	0.059	0.472	0.052
			0.3	0.782	0.763	0.728	0.737	0.085	0.745	0.084	0.721	0.065	0.694	0.057	0.605	0.597	0.277	0.623	0.119	0.625	0.122	0.634	0.063	0.644	0.054
			0.4	0.921	0.890	0.869	0.880	0.106	0.884	0.111	0.872	0.070	0.842	0.058	0.735	0.719	0.361	0.745	0.162	0.731	0.168	0.753	0.063	0.753	0.056
			0.5	0.978	0.959	0.947	0.945	0.128	0.951	0.130	0.948	0.076	0.929	0.059	0.815	0.790	0.439	0.810	0.201	0.813	0.205	0.825	0.060	0.823	0.053
		$[1, t]'$	0.1	0.172	0.177	0.128	0.166	0.057	0.162	0.056	0.152	0.063	0.165	0.056	0.122	0.133	0.050	0.168	0.072	0.158	0.079	0.182	0.059	0.165	0.052
			0.2	0.394	0.392	0.311	0.376	0.077	0.389	0.077	0.371	0.068	0.388	0.057	0.300	0.311	0.121	0.390	0.129	0.369	0.126	0.409	0.062	0.346	0.053
			0.3	0.631	0.610	0.521	0.596	0.110	0.598	0.095	0.586	0.071	0.619	0.059	0.513	0.512	0.223	0.593	0.190	0.579	0.195	0.618	0.065	0.515	0.053
			0.4	0.802	0.765	0.684	0.761	0.141	0.770	0.137	0.760	0.078	0.783	0.056	0.682	0.664	0.311	0.726	0.265	0.730	0.255	0.733	0.063	0.642	0.057
			0.5	0.915	0.877	0.818	0.864	0.172	0.873	0.175	0.871	0.087	0.884	0.056	0.792	0.764	0.395	0.821	0.314	0.819	0.318	0.821	0.075	0.728	0.055

Note: Simulations are based on DGP (a). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 0$ and $p = 2$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 0$ and $p = 2$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are $Haar$ and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, ρ_c demonstrates the AR(1) coefficient of the process $z_{2,t}$.

Table 7: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 0$ with Sample Size=1024

T	θ	δ_t	ρ_c	$d_1=0.1$: Λ	Haar Λ^L	Haar Λ^{L*}	D_2 Λ^L	D_2 Λ^{L*}	S_2 Λ^L	S_2 Λ^{L*}	S_4 Λ^L	S_4 Λ^{L*}	S_8 Λ^L	S_8 Λ^{L*}	$d_1=1$: Λ	Haar Λ^L	Haar Λ^{L*}	D_2 Λ^L	D_2 Λ^{L*}	S_2 Λ^L	S_2 Λ^{L*}	S_4 Λ^L	S_4 Λ^{L*}	S_8 Λ^L	S_8 Λ^{L*}
1024	-0.9	0	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.597	1.000	0.999	0.972	0.999	0.961	1.000	0.969	0.999	0.932	1.000	0.524
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.884	1.000	1.000	0.999	1.000	0.999	1.000	0.999	1.000	0.998	1.000	0.845
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.950
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.985
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994
		1	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.525	1.000	0.114	1.000	1.000	0.997	1.000	0.996	1.000	0.994	1.000	0.294	1.000	0.032
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.757	1.000	0.204	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.664	1.000	0.116
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.868	1.000	0.261	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.863	1.000	0.221
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.929	1.000	0.293	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.944	1.000	0.384
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.956	1.000	0.306	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.966	1.000	0.500
		$[1, t]'$	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.590	1.000	0.133	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.377	1.000	0.043
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.806	1.000	0.244	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.779	1.000	0.138
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.905	1.000	0.305	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.936	1.000	0.280
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.955	1.000	0.351	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.977	1.000	0.440
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.949	1.000	0.376	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.991	1.000	0.520
	-0.5	0	0.1	1.000	1.000	1.000	1.000	0.995	1.000	0.994	1.000	0.531	1.000	0.181	0.919	0.883	0.544	0.882	0.457	0.889	0.447	0.884	0.397	0.886	0.176
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.835	1.000	0.297	0.991	0.977	0.800	0.976	0.753	0.979	0.750	0.977	0.593	0.978	0.249
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	1.000	0.366	0.999	0.995	0.905	0.995	0.882	0.994	0.888	0.993	0.774	0.993	0.358
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.481	1.000	0.998	0.950	0.998	0.937	0.999	0.943	0.998	0.876	0.998	0.442
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.627	1.000	1.000	0.970	1.000	0.967	0.999	0.965	1.000	0.936	1.000	0.498
		1	0.1	1.000	1.000	1.000	1.000	0.641	1.000	0.645	1.000	0.078	1.000	0.033	0.978	0.959	0.591	0.957	0.556	0.964	0.557	0.963	0.056	0.963	0.039
			0.2	1.000	1.000	1.000	1.000	0.998	1.000	0.999	1.000	0.185	1.000	0.044	0.999	0.997	0.876	0.997	0.859	0.997	0.855	0.998	0.112	0.999	0.030
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.296	1.000	0.063	1.000	1.000	0.960	1.000	0.957	1.000	0.953	1.000	0.176	1.000	0.029
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.428	1.000	0.086	1.000	1.000	0.983	1.000	0.985	1.000	0.986	1.000	0.230	1.000	0.028
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.540	1.000	0.114	1.000	1.000	0.995	1.000	0.996	1.000	0.995	1.000	0.287	1.000	0.035
		$[1, t]'$	0.1	1.000	1.000	1.000	1.000	0.682	1.000	0.666	1.000	0.085	1.000	0.033	0.988	0.970	0.595	0.973	0.662	0.977	0.675	0.969	0.078	0.978	0.043
			0.2	1.000	1.000	1.000	1.000	0.998	1.000	0.997	1.000	0.210	1.000	0.047	1.000	1.000	0.911	1.000	0.951	1.000	0.952	1.000	0.159	1.000	0.036
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.352	1.000	0.070	1.000	1.000	0.983	1.000	0.994	1.000	0.995	1.000	0.243	1.000	0.037
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.470	1.000	0.100	1.000	1.000	0.997	1.000	1.000	1.000	0.999	1.000	0.314	1.000	0.037
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.573	1.000	0.138	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.422	1.000	0.046
	0	0	0.1	1.000	1.000	1.000	1.000	0.934	1.000	0.927	1.000	0.476	1.000	0.180	0.857	0.847	0.525	0.845	0.397	0.843	0.388	0.849	0.303	0.839	0.156
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.727	1.000	0.255	0.962	0.953	0.730	0.950	0.620	0.952	0.637	0.954	0.490	0.950	0.227
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.906	1.000	0.311	0.984	0.977	0.816	0.980	0.771	0.980	0.766	0.978	0.600	0.980	0.277
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.978	1.000	0.362	0.994	0.990	0.871	0.990	0.838	0.991	0.837	0.991	0.680	0.988	0.318
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994	1.000	0.440	0.998	0.995	0.912	0.995	0.895	0.994	0.899	0.997	0.780	0.995	0.353
		1	0.1	1.000	1.000	1.000	1.000	0.342	1.000	0.346	1.000	0.089	1.000	0.060	0.943	0.936	0.564	0.939	0.410	0.940	0.414	0.942	0.061	0.943	0.058
			0.2	1.000	1.000	1.000	1.000	0.867	1.000	0.828	1.000	0.151	1.000	0.061	0.993	0.990	0.803	0.992	0.703	0.992	0.693	0.993	0.098	0.992	0.049
			0.3	1.000	1.000	1.000	1.000	0.994	1.000	0.994	1.000	0.256	1.000	0.066	0.999	0.998	0.906	0.998	0.837	0.998	0.840	0.998	0.138	0.998	0.044
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.313	1.000	0.079	1.000	0.999	0.941	0.999	0.906	1.000	0.921	1.000	0.172	1.000	0.038
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.379	1.000	0.091	1.000	1.000	0.963	1.000	0.947	1.000	0.957	1.000	0.200	1.000	0.039
		$[1, t]'$	0.1	0.999	0.999	0.995	0.999	0.433	1.000	0.422	1.000	0.098	1.000	0.062	0.947	0.939	0.572	0.947	0.519	0.951	0.533	0.945	0.086	0.954	0.064
			0.2	1.000	1.000	1.000	1.000	0.877	1.000	0.884	1.000	0.186	1.000	0.062	0.998	0.996	0.838	0.997	0.838	0.998	0.830	0.997	0.145	0.998	0.056
			0.3	1.000	1.000	1.000	1.000	0.993	1.000	0.994	1.000	0.283	1.000	0.070	1.000	1.000	0.942	1.000	0.945	1.000	0.939	1.000	0.194	1.000	0.048
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.369	1.000	0.086	1.000	1.000	0.975	1.000	0.980	1.000	0.975	1.000	0.237	1.000	0.050
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.461	1.000	0.101	1.000	1.000	0.989	1.000	0.993	1.000	0.994	1.000	0.293	1.000	0.047
	0.5	0	0.1	1.000	1.000	1.000	1.000	0.910	1.000	0.916	1.000	0.474	1.000	0.187	0.846	0.842	0.527	0.846	0.389	0.842	0.388	0.838	0.287	0.838	0.142
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.705	1.000	0.278	0.955	0.951	0.718	0.949	0.608	0.947	0.621	0.949	0.434	0.947	0.194
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.875	1.000	0.313	0.981	0.978	0.812	0.978	0.732	0.978	0.738	0.978	0.566	0.972	0.266
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.951	1.000	0.380	0.991	0.988	0.859	0.990	0.814	0.988	0.818	0.987	0.628	0.986	0.295
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	1.000	0.434	0.995	0.992	0.891	0.992	0.854	0.993	0.861	0.993	0.671	0.993	0.326
		1	0.1	1.000	1.000	1.000	1.000	0.317	1.000	0.307	1.000	0.093	1.000	0.065	0.937	0.934	0.581	0.929	0.376	0.931	0.377	0.935	0.064	0.945	0.057
			0.2	1.000	1.000	1.000	1.000	0.790	1.000	0.748	1.000	0.163	1.000	0.065	0.991	0.989	0.807	0.991	0.661	0.989	0.670	0.990	0.100	0.991	0.053
			0.3	1.000	1.000	1.000	1.000	0.975	1.000	0.978	1.000	0.230	1.000	0.072	0.998	0.998	0.902	0.998	0.811	0.998	0.798	0.998	0.129	0.998	0.049
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.310	1.000	0.085	0.999	0.998	0.924	0.999	0.879	1.000	0.878	0.999	0.155	0.999	0.041
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.370	1.000	0.092	1.000	1.000	0.953	1.000	0.918	1.000	0.914	1.000	0.194	1.000	0.043
		$[1, t]'$	0.1	0.999	0.999	0.995	0.999	0.381	0.998	0.400	0.999	0.097	0.999	0.069	0.936	0.932	0.607	0.941	0.488	0.947	0.481	0.939	0.083	0.948	0.067
			0.2	1.000	1.000	1.000	1.000	0.809	1.000	0.806	1.000	0.186	1.000	0.072	0.997	0.996	0.851	0.996	0.807	0.997	0.792	0.996	0.143	0.998	0.056
			0.3	1.000	1.000	1.000	1.000	0.978	1.000	0.976	1.000	0.281	1.000	0.083	1.000	1.000	0.937	1.000	0.912	1.000	0.914	1.000	0.202	1.000	0.055
			0.4	1.000	1.000	1.000	1.000	0.997	1.000	0.998	1.000	0.362	1.000	0.094	1.000	1.000	0.964	1.000	0.962	1.000	0.955	1.000	0.239	1.000	0.051
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.435	1.000	0.102	1.000	1.000	0.981	1.000	0.980	1.000	0.981	1.000	0.271	1.000	0.048
	0.9	0	0.1	1.000	1.000	1.000	1.000	0.906	1.000	0.914	1.000	0.450	1.000	0.196	0.839	0.834	0.520	0.844	0.393	0.842	0.395	0.832	0.283	0.839	0.149
			0.2	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.725	1.000	0.275	0.952	0.948	0.714	0.950	0.619	0.948	0.608	0.948	0.447	0.949	0.227
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.890	1.000	0.318	0.980	0.977	0.811	0.975	0.732	0.974	0.725	0.976	0.530	0.975	0.252
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.946	1.000	0.385	0.988	0.986	0.868	0.987	0.800	0.984	0.810	0.987	0.604	0.987	0.314
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.980	1.000	0.430	0.993	0.990	0.889	0.993	0.849	0.993	0.849	0.992	0.667	0.992	0.330
		1	0.1	1.000	1.000	1.000	1.000	0.309	1.000	0.312	1.000	0.091	1.000	0.067	0.931	0.929	0.586	0.932	0.378	0.933	0.377	0.943	0.065	0.939	0.060
			0.2	1.000	1.000	1.000	1.000	0.745	1.000	0.752	1.000	0.159	1.000	0.065	0.992	0.991	0.816	0.989	0.656	0.990	0.663	0.991	0.093	0.991	0.052
			0.3	1.000	1.000	1.000	1.000	0.959	1.000	0.968	1.000	0.227	1.000	0.074	0.998	0.997	0.885	0.998	0.772	0.998	0.787	0.997	0.127	0.998	0.045
			0.4	1.000	1.000	1.000	1.000	0.998	1.000	0.998	1.000	0.297	1.000	0.086	1.000	0.999	0.934	0.999	0.865	0.999	0.857	0.999	0.157	1.000	0.042
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.367	1.000	0.093	1.000	1.000	0.958	1.000	0.908	1.000	0.914	1.000	0.184	1.000	0.044
		$[1, t]'$	0.1	0.999	0.999	0.995	0.999	0.369	0.999	0.375	0.999	0.099	0.999	0.069	0.933	0.930	0.567	0.942	0.490	0.944	0.479	0.940	0.089	0.952	0.067
			0.2	1.000	1.000	1.000	1.000	0.790	1.000	0.806	1.000	0.178	1.000	0.070	0.995	0.995	0.853	0.996	0.780	0.997	0.788	0.997	0.140	0.997	0.060
			0.3	1.000	1.000	1.000	1.000	0.970	1.000	0.976	1.000	0.274	1.000	0.082	1.000	0.999	0.928	1.000	0.906	1.000	0.910	1.000	0.188	1.000	0.052
			0.4	1.000	1.000	1.000	1.000	0.999	1.000	0.997	1.000	0.358	1.000	0.098	1.000	1.000	0.968	1.000	0.956	1.000	0.949	1.000	0.233	1.000	0.050
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.430	1.000	0.110	1.000	1.000	0.982	1.000	0.977	1.000	0.979	1.000	0.256	1.000	0.051

Note: Simulations are based on DGP (a). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 0$ and $p = 2$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 0$ and $p = 2$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are $Haar$ and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, ρ_c demonstrates the AR(1) coefficient of the process $z_{2,t}$.

Table 8: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 1$ with Sample Size=128

				$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}
T	θ	δ_t	ρ_c		Haar		D_2		S_2		S_4		S_8			Haar		D_2		S_2		S_4		S_8	
128	-0.9	0	0.1	0.320	0.341	0.644	1.000	0.469	1.000	0.441	0.999	0.194	0.996	0.148	0.922	0.756	0.401	0.701	0.104	0.700	0.100	0.676	0.038	0.566	0.037
			0.2	0.381	0.421	0.769	1.000	0.573	1.000	0.546	1.000	0.235	0.999	0.191	0.962	0.859	0.573	0.821	0.220	0.814	0.214	0.803	0.068	0.715	0.067
			0.3	0.399	0.441	0.795	1.000	0.587	1.000	0.598	1.000	0.237	0.999	0.188	0.965	0.880	0.640	0.848	0.272	0.847	0.270	0.829	0.100	0.747	0.101
			0.4	0.407	0.456	0.806	1.000	0.583	1.000	0.594	1.000	0.241	0.999	0.195	0.966	0.886	0.631	0.851	0.298	0.858	0.290	0.842	0.122	0.766	0.117
			0.5	0.411	0.456	0.808	1.000	0.568	1.000	0.598	1.000	0.237	0.999	0.197	0.969	0.894	0.663	0.855	0.307	0.858	0.320	0.848	0.133	0.776	0.122
		1	0.1	0.179	0.199	0.611	1.000	0.142	0.999	0.140	0.999	0.038	0.999	0.024	0.987	0.899	0.516	0.895	0.131	0.889	0.136	0.890	0.008	0.887	0.010
			0.2	0.240	0.275	0.731	1.000	0.180	1.000	0.181	1.000	0.046	1.000	0.025	0.995	0.955	0.677	0.948	0.235	0.941	0.234	0.943	0.013	0.935	0.010
			0.3	0.260	0.300	0.759	1.000	0.195	1.000	0.200	1.000	0.053	1.000	0.028	0.996	0.960	0.723	0.954	0.268	0.952	0.293	0.951	0.021	0.942	0.012
			0.4	0.273	0.320	0.774	1.000	0.198	1.000	0.209	1.000	0.054	1.000	0.029	0.996	0.961	0.729	0.954	0.306	0.952	0.309	0.954	0.026	0.947	0.013
			0.5	0.265	0.305	0.766	1.000	0.192	1.000	0.202	1.000	0.050	1.000	0.029	0.995	0.964	0.750	0.962	0.313	0.960	0.314	0.956	0.030	0.951	0.016
		$[1, t]'$	0.1	0.108	0.119	0.476	0.996	0.164	0.997	0.173	0.993	0.038	0.996	0.019	0.997	0.930	0.471	0.896	0.197	0.903	0.206	0.891	0.008	0.813	0.006
			0.2	0.144	0.164	0.574	0.998	0.217	0.998	0.236	0.996	0.045	0.997	0.020	0.999	0.970	0.698	0.953	0.308	0.953	0.300	0.937	0.011	0.887	0.006
			0.3	0.161	0.185	0.608	0.999	0.236	0.999	0.259	0.998	0.048	0.997	0.025	0.999	0.971	0.724	0.959	0.364	0.959	0.368	0.946	0.018	0.908	0.007
			0.4	0.170	0.200	0.631	0.999	0.265	0.999	0.251	0.998	0.050	0.999	0.028	0.999	0.978	0.756	0.961	0.377	0.964	0.390	0.952	0.023	0.917	0.009
			0.5	0.169	0.200	0.633	0.999	0.254	0.999	0.257	0.998	0.055	0.998	0.026	0.999	0.977	0.749	0.963	0.407	0.965	0.384	0.955	0.024	0.922	0.011
	-0.5	0	0.1	0.546	0.439	0.288	0.500	0.102	0.481	0.099	0.437	0.068	0.376	0.062	0.308	0.208	0.047	0.175	0.022	0.177	0.021	0.163	0.025	0.111	0.032
			0.2	0.941	0.822	0.656	0.877	0.205	0.854	0.219	0.822	0.095	0.766	0.097	0.605	0.427	0.139	0.393	0.034	0.390	0.036	0.363	0.026	0.271	0.033
			0.3	0.999	0.973	0.902	0.983	0.324	0.982	0.324	0.974	0.137	0.945	0.124	0.781	0.588	0.230	0.539	0.043	0.536	0.047	0.519	0.025	0.401	0.035
			0.4	1.000	0.997	0.979	0.998	0.402	0.998	0.411	0.996	0.178	0.991	0.138	0.875	0.687	0.333	0.640	0.067	0.651	0.077	0.615	0.029	0.498	0.032
			0.5	1.000	1.000	0.994	1.000	0.458	1.000	0.497	1.000	0.187	0.997	0.149	0.915	0.755	0.405	0.695	0.096	0.698	0.093	0.670	0.034	0.560	0.034
		1	0.1	0.444	0.370	0.198	0.488	0.026	0.481	0.025	0.437	0.028	0.449	0.027	0.403	0.273	0.053	0.302	0.022	0.277	0.021	0.293	0.021	0.308	0.028
			0.2	0.876	0.745	0.518	0.849	0.051	0.844	0.048	0.825	0.027	0.833	0.025	0.750	0.563	0.150	0.588	0.040	0.567	0.039	0.572	0.016	0.582	0.020
			0.3	0.994	0.941	0.787	0.978	0.075	0.973	0.079	0.967	0.032	0.968	0.022	0.912	0.747	0.260	0.767	0.063	0.750	0.063	0.765	0.012	0.757	0.016
			0.4	1.000	0.990	0.930	0.997	0.106	0.998	0.111	0.996	0.034	0.996	0.025	0.966	0.850	0.420	0.850	0.087	0.845	0.098	0.846	0.008	0.840	0.013
			0.5	1.000	0.997	0.971	1.000	0.135	1.000	0.145	0.999	0.035	0.999	0.021	0.985	0.901	0.485	0.895	0.121	0.887	0.131	0.886	0.009	0.883	0.011
		$[1, t]'$	0.1	0.262	0.218	0.084	0.406	0.031	0.408	0.030	0.351	0.022	0.368	0.025	0.381	0.235	0.037	0.241	0.043	0.254	0.037	0.263	0.019	0.240	0.024
			0.2	0.659	0.523	0.269	0.761	0.058	0.760	0.056	0.706	0.024	0.738	0.021	0.758	0.540	0.115	0.531	0.081	0.547	0.083	0.560	0.012	0.478	0.016
			0.3	0.926	0.785	0.517	0.938	0.092	0.938	0.097	0.915	0.029	0.921	0.017	0.935	0.757	0.249	0.739	0.118	0.755	0.116	0.741	0.010	0.658	0.011
			0.4	0.992	0.910	0.689	0.982	0.137	0.989	0.142	0.975	0.032	0.979	0.019	0.986	0.875	0.373	0.846	0.149	0.859	0.164	0.837	0.010	0.752	0.008
			0.5	0.999	0.965	0.805	0.996	0.177	0.996	0.173	0.990	0.032	0.994	0.019	0.997	0.932	0.491	0.898	0.215	0.901	0.185	0.888	0.010	0.815	0.008
	0	0	0.1	0.334	0.313	0.260	0.311	0.090	0.295	0.086	0.264	0.053	0.219	0.056	0.158	0.133	0.058	0.127	0.032	0.121	0.035	0.110	0.035	0.068	0.031
			0.2	0.706	0.631	0.554	0.615	0.158	0.599	0.153	0.553	0.083	0.479	0.073	0.363	0.300	0.120	0.271	0.046	0.272	0.043	0.255	0.038	0.177	0.034
			0.3	0.926	0.840	0.779	0.841	0.234	0.826	0.219	0.792	0.101	0.712	0.092	0.525	0.423	0.155	0.394	0.056	0.401	0.063	0.374	0.033	0.269	0.034
			0.4	0.990	0.946	0.914	0.946	0.281	0.941	0.275	0.919	0.130	0.863	0.108	0.661	0.535	0.226	0.483	0.056	0.484	0.057	0.461	0.032	0.343	0.031
			0.5	0.999	0.985	0.970	0.983	0.334	0.977	0.361	0.970	0.144	0.941	0.121	0.735	0.595	0.249	0.545	0.067	0.556	0.072	0.526	0.034	0.410	0.034
		1	0.1	0.289	0.266	0.190	0.260	0.029	0.253	0.032	0.230	0.042	0.242	0.040	0.189	0.165	0.046	0.195	0.030	0.179	0.030	0.200	0.037	0.208	0.034
			0.2	0.649	0.569	0.446	0.561	0.039	0.545	0.040	0.520	0.040	0.532	0.037	0.444	0.372	0.106	0.412	0.042	0.397	0.043	0.409	0.031	0.422	0.032
			0.3	0.897	0.796	0.681	0.789	0.052	0.786	0.057	0.755	0.042	0.770	0.037	0.668	0.561	0.174	0.596	0.054	0.562	0.056	0.586	0.029	0.595	0.030
			0.4	0.981	0.919	0.848	0.916	0.071	0.913	0.069	0.888	0.040	0.901	0.032	0.798	0.681	0.259	0.714	0.070	0.695	0.074	0.698	0.021	0.699	0.025
			0.5	0.998	0.973	0.933	0.969	0.084	0.969	0.090	0.958	0.043	0.962	0.030	0.872	0.754	0.292	0.781	0.082	0.744	0.086	0.761	0.019	0.767	0.021
		$[1, t]'$	0.1	0.188	0.177	0.078	0.172	0.032	0.189	0.032	0.153	0.039	0.168	0.037	0.145	0.128	0.030	0.137	0.044	0.151	0.047	0.153	0.036	0.140	0.032
			0.2	0.480	0.418	0.233	0.421	0.044	0.439	0.048	0.386	0.037	0.398	0.034	0.372	0.314	0.076	0.331	0.072	0.348	0.070	0.372	0.033	0.310	0.028
			0.3	0.770	0.669	0.437	0.648	0.064	0.661	0.074	0.611	0.038	0.639	0.032	0.609	0.504	0.123	0.528	0.103	0.553	0.106	0.546	0.023	0.462	0.023
			0.4	0.928	0.824	0.621	0.822	0.086	0.827	0.079	0.792	0.042	0.803	0.031	0.792	0.662	0.203	0.668	0.126	0.681	0.137	0.672	0.018	0.580	0.021
			0.5	0.987	0.918	0.752	0.911	0.098	0.921	0.105	0.888	0.039	0.904	0.025	0.889	0.761	0.259	0.757	0.157	0.769	0.168	0.748	0.015	0.655	0.017
	0.5	0	0.1	0.300	0.298	0.267	0.290	0.097	0.273	0.085	0.239	0.055	0.197	0.054	0.144	0.129	0.054	0.114	0.034	0.119	0.040	0.102	0.040	0.062	0.034
			0.2	0.611	0.583	0.539	0.564	0.167	0.556	0.151	0.505	0.079	0.433	0.065	0.312	0.274	0.111	0.252	0.052	0.265	0.060	0.246	0.036	0.162	0.034
			0.3	0.850	0.803	0.766	0.789	0.223	0.774	0.236	0.729	0.105	0.652	0.079	0.471	0.406	0.170	0.369	0.069	0.377	0.071	0.357	0.034	0.256	0.034
			0.4	0.955	0.913	0.889	0.905	0.280	0.898	0.276	0.878	0.126	0.808	0.088	0.582	0.493	0.223	0.468	0.083	0.462	0.083	0.433	0.035	0.325	0.030
			0.5	0.989	0.968	0.955	0.964	0.313	0.954	0.336	0.944	0.131	0.899	0.108	0.656	0.553	0.235	0.524	0.077	0.513	0.080	0.502	0.037	0.379	0.029
		1	0.1	0.253	0.258	0.188	0.226	0.034	0.228	0.033	0.209	0.044	0.213	0.043	0.165	0.157	0.050	0.179	0.037	0.163	0.037	0.182	0.042	0.199	0.036
			0.2	0.558	0.545	0.436	0.503	0.047	0.498	0.045	0.460	0.049	0.475	0.043	0.386	0.352	0.105	0.392	0.044	0.371	0.051	0.394	0.036	0.410	0.033
			0.3	0.794	0.764	0.664	0.733	0.066	0.709	0.059	0.689	0.048	0.712	0.043	0.572	0.516	0.183	0.553	0.066	0.540	0.059	0.559	0.031	0.565	0.030
			0.4	0.925	0.886	0.808	0.873	0.075	0.857	0.076	0.840	0.051	0.853	0.039	0.709	0.634	0.228	0.682	0.082	0.647	0.070	0.666	0.027	0.666	0.027
			0.5	0.983	0.958	0.913	0.939	0.091	0.940	0.091	0.926	0.052	0.933	0.039	0.799	0.714	0.287	0.746	0.096	0.720	0.092	0.739	0.023	0.739	0.025
		$[1, t]'$	0.1	0.184	0.180	0.090	0.156	0.040	0.163	0.036	0.129	0.041	0.133	0.039	0.117	0.114	0.037	0.117	0.051	0.134	0.051	0.147	0.040	0.128	0.032
			0.2	0.404	0.387	0.231	0.372	0.051	0.378	0.052	0.329	0.041	0.345	0.038	0.300	0.283	0.083	0.317	0.083	0.324	0.075	0.344	0.034	0.290	0.028
			0.3	0.654	0.613	0.424	0.574	0.074	0.602	0.071	0.535	0.046	0.570	0.038	0.511	0.463	0.142	0.499	0.124	0.503	0.106	0.511	0.030	0.426	0.026
			0.4	0.827	0.766	0.583	0.743	0.094	0.753	0.093	0.710	0.050	0.739	0.036	0.676	0.605	0.190	0.631	0.145	0.646	0.136	0.640	0.028	0.543	0.021
			0.5	0.931	0.874	0.722	0.849	0.107	0.861	0.113	0.826	0.048	0.844	0.035	0.785	0.702	0.255	0.716	0.155	0.732	0.154	0.716	0.021	0.614	0.021
	0.9	0	0.1	0.302	0.310	0.259	0.281	0.099	0.272	0.102	0.240	0.065	0.195	0.048	0.151	0.138	0.060	0.114	0.044	0.118	0.046	0.104	0.036	0.062	0.031
			0.2	0.616	0.605	0.527	0.569	0.159	0.540	0.155	0.502	0.078	0.440	0.063	0.326	0.288	0.114	0.256	0.055	0.250	0.050	0.238	0.035	0.166	0.032
			0.3	0.846	0.818	0.752	0.786	0.202	0.767	0.218	0.737	0.097	0.655	0.079	0.469	0.404	0.170	0.363	0.063	0.370	0.071	0.353	0.031	0.245	0.029
			0.4	0.948	0.921	0.875	0.900	0.267	0.891	0.270	0.866	0.117	0.802	0.090	0.568	0.486	0.198	0.458	0.087	0.454	0.081	0.436	0.035	0.325	0.033
			0.5	0.987	0.967	0.945	0.958	0.324	0.953	0.328	0.936	0.137	0.894	0.094	0.652	0.566	0.241	0.522	0.083	0.520	0.087	0.499	0.036	0.378	0.031
		1	0.1	0.246	0.245	0.188	0.223	0.035	0.220	0.036	0.192	0.050	0.200	0.042	0.157	0.149	0.049	0.177	0.037	0.166	0.040	0.182	0.040	0.192	0.037
			0.2	0.527	0.515	0.421	0.484	0.042	0.486	0.045	0.452	0.052	0.470	0.044	0.380	0.350	0.113	0.396	0.049	0.365	0.048	0.373	0.033	0.403	0.031
			0.3	0.771	0.739	0.641	0.724	0.061	0.711	0.059	0.689	0.052	0.692	0.042	0.570	0.516	0.171	0.567	0.066	0.527	0.064	0.548	0.029	0.558	0.027
			0.4	0.913	0.874	0.803	0.861	0.082	0.859	0.078	0.841	0.050	0.847	0.040	0.700	0.629	0.240	0.670	0.072	0.648	0.083	0.659	0.025	0.675	0.023
			0.5	0.974	0.946	0.900	0.939	0.089	0.934	0.086	0.920	0.053	0.928	0.038	0.781	0.703	0.288	0.733	0.086	0.709	0.089	0.730	0.022	0.735	0.023
		$[1, t]'$	0.1	0.174	0.175	0.096	0.152	0.039	0.148	0.032	0.128	0.045	0.134	0.041	0.114	0.114	0.035	0.128	0.052	0.128	0.049	0.144	0.040	0.126	0.033
			0.2	0.390	0.384	0.244	0.352	0.050	0.371	0.053	0.314	0.045	0.347	0.037	0.289	0.275	0.085	0.314	0.081	0.319	0.081	0.340	0.031	0.288	0.027
			0.3	0.627	0.602	0.426	0.571	0.066	0.585	0.069	0.544	0.046	0.556	0.038	0.496	0.457	0.136	0.492	0.111	0.510	0.108	0.510	0.029	0.428	0.024
			0.4	0.806	0.760	0.599	0.731	0.094	0.748	0.094	0.704	0.053	0.726	0.036	0.649	0.590	0.188	0.636	0.145	0.644	0.135	0.642	0.027	0.536	0.021
			0.5	0.911	0.862	0.720	0.846	0.108	0.853	0.110	0.817	0.053	0.842	0.035	0.774	0.698	0.263	0.730	0.162	0.735	0.165	0.730	0.023	0.596	0.019

Note: Simulations are based on DGP (b). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, ρ_c demonstrates the AR(1) coefficient of the process $z_{2,t}$.

Table 9: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 1$ with Sample Size=1024

T	θ	δ_t	ρ_c	$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
1024	-0.9	0	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.542	1.000	0.998	0.922	0.998	0.906	0.997	0.915	0.998	0.825	0.997	0.301
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.583	1.000	0.999	0.964	0.999	0.959	1.000	0.959	0.999	0.916	0.999	0.451
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.636	1.000	1.000	0.966	1.000	0.962	1.000	0.965	0.999	0.915	1.000	0.492
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.629	1.000	0.999	0.968	0.999	0.968	1.000	0.960	0.999	0.922	1.000	0.492
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.600	1.000	1.000	0.973	0.999	0.967	1.000	0.963	0.999	0.929	0.999	0.494
		1	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.444	1.000	0.086	1.000	1.000	0.984	1.000	0.984	1.000	0.980	1.000	0.134	1.000	0.008
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.504	1.000	0.111	1.000	1.000	0.994	1.000	0.994	1.000	0.995	1.000	0.248	1.000	0.019
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.489	1.000	0.115	1.000	1.000	0.995	1.000	0.994	1.000	0.995	1.000	0.272	1.000	0.024
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.534	1.000	0.109	1.000	1.000	0.993	1.000	0.994	1.000	0.993	1.000	0.274	1.000	0.032
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.532	1.000	0.110	1.000	1.000	0.994	1.000	0.996	1.000	0.996	1.000	0.275	1.000	0.028
		$[1, t]'$	0.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.471	1.000	0.099	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.220	1.000	0.016
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.560	1.000	0.123	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.336	1.000	0.025
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.541	1.000	0.125	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.391	1.000	0.034
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.575	1.000	0.127	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.380	1.000	0.039
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.564	1.000	0.128	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.359	1.000	0.045
	-0.5	0	0.1	1.000	1.000	1.000	1.000	0.995	1.000	0.995	1.000	0.510	1.000	0.176	0.910	0.870	0.508	0.871	0.406	0.876	0.378	0.873	0.326	0.864	0.132
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.867	1.000	0.279	0.986	0.967	0.736	0.968	0.681	0.969	0.681	0.969	0.597	0.968	0.155
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.985	1.000	0.406	0.998	0.991	0.847	0.990	0.813	0.990	0.816	0.988	0.723	0.990	0.200
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.468	1.000	0.996	0.904	0.996	0.879	0.995	0.875	0.995	0.833	0.995	0.257
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.513	1.000	0.998	0.920	0.998	0.914	0.997	0.914	0.998	0.815	0.997	0.288
		1	0.1	1.000	1.000	1.000	1.000	0.588	1.000	0.576	1.000	0.076	1.000	0.032	0.977	0.959	0.554	0.961	0.510	0.957	0.500	0.963	0.045	0.962	0.034
			0.2	1.000	1.000	1.000	1.000	0.994	1.000	0.994	1.000	0.177	1.000	0.037	0.999	0.997	0.855	0.997	0.831	0.998	0.840	0.998	0.081	0.997	0.020
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.289	1.000	0.059	1.000	1.000	0.935	1.000	0.933	1.000	0.934	1.000	0.104	1.000	0.014
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.386	1.000	0.078	1.000	1.000	0.971	1.000	0.970	1.000	0.972	1.000	0.115	1.000	0.009
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.428	1.000	0.087	1.000	1.000	0.979	1.000	0.985	1.000	0.985	1.000	0.136	1.000	0.010
		$[1, t]'$	0.1	1.000	1.000	0.998	1.000	0.647	1.000	0.617	1.000	0.079	1.000	0.031	0.986	0.966	0.556	0.969	0.668	0.975	0.661	0.974	0.079	0.976	0.036
			0.2	1.000	1.000	1.000	1.000	0.996	1.000	0.990	1.000	0.190	1.000	0.039	1.000	1.000	0.889	1.000	0.933	1.000	0.942	1.000	0.136	1.000	0.028
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.316	1.000	0.061	1.000	1.000	0.971	1.000	0.992	1.000	0.993	1.000	0.174	1.000	0.016
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.402	1.000	0.086	1.000	1.000	0.993	1.000	0.999	1.000	1.000	1.000	0.194	1.000	0.015
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.494	1.000	0.100	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.221	1.000	0.016
	0	$[1, t]'$	0.1	1.000	1.000	1.000	1.000	0.934	1.000	0.940	1.000	0.461	1.000	0.195	0.852	0.835	0.505	0.828	0.337	0.838	0.362	0.828	0.267	0.826	0.136
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.710	1.000	0.256	0.957	0.944	0.671	0.946	0.568	0.944	0.571	0.944	0.373	0.936	0.157
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.892	1.000	0.324	0.985	0.975	0.777	0.971	0.709	0.974	0.695	0.974	0.516	0.973	0.196
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.980	1.000	0.354	0.992	0.985	0.824	0.987	0.777	0.985	0.772	0.985	0.559	0.985	0.200
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	1.000	0.416	0.998	0.992	0.859	0.990	0.819	0.991	0.825	0.990	0.707	0.991	0.200
		1	0.1	1.000	1.000	1.000	1.000	0.311	1.000	0.306	1.000	0.083	1.000	0.055	0.940	0.930	0.546	0.936	0.403	0.928	0.381	0.939	0.053	0.944	0.051
			0.2	1.000	1.000	1.000	1.000	0.812	1.000	0.863	1.000	0.154	1.000	0.053	0.993	0.990	0.797	0.991	0.687	0.988	0.669	0.992	0.076	0.993	0.035
			0.3	1.000	1.000	1.000	1.000	0.988	1.000	0.982	1.000	0.238	1.000	0.063	0.999	0.997	0.873	0.997	0.809	0.998	0.807	0.997	0.093	0.998	0.024
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.292	1.000	0.072	1.000	0.999	0.922	0.999	0.889	0.999	0.891	0.999	0.114	1.000	0.020
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.373	1.000	0.085	1.000	1.000	0.946	1.000	0.937	1.000	0.930	1.000	0.120	1.000	0.018
		$[1, t]'$	0.1	0.999	0.999	0.994	0.999	0.388	0.999	0.376	0.999	0.084	0.999	0.057	0.950	0.942	0.562	0.939	0.491	0.942	0.491	0.952	0.081	0.959	0.059
			0.2	1.000	1.000	1.000	1.000	0.835	1.000	0.856	1.000	0.162	1.000	0.056	0.998	0.996	0.837	0.997	0.808	0.997	0.820	0.998	0.120	0.997	0.043
			0.3	1.000	1.000	1.000	1.000	0.980	1.000	0.988	1.000	0.260	1.000	0.063	1.000	1.000	0.925	1.000	0.930	1.000	0.927	1.000	0.162	1.000	0.029
			0.4	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.332	1.000	0.077	1.000	1.000	0.959	1.000	0.973	1.000	0.975	1.000	0.181	1.000	0.027
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.439	1.000	0.088	1.000	1.000	0.979	1.000	0.989	1.000	0.990	1.000	0.194	1.000	0.023
	0.5	0	0.1	1.000	1.000	1.000	1.000	0.902	1.000	0.903	1.000	0.470	1.000	0.181	0.836	0.826	0.480	0.831	0.335	0.830	0.332	0.822	0.254	0.818	0.116
			0.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.716	1.000	0.264	0.945	0.936	0.692	0.940	0.549	0.944	0.571	0.936	0.375	0.940	0.168
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.877	1.000	0.309	0.977	0.969	0.767	0.975	0.682	0.972	0.694	0.969	0.455	0.972	0.167
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.967	1.000	0.352	0.986	0.979	0.811	0.983	0.718	0.982	0.728	0.982	0.541	0.983	0.184
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.987	1.000	0.408	0.993	0.988	0.847	0.989	0.792	0.988	0.788	0.988	0.584	0.988	0.172
		1	0.1	1.000	1.000	1.000	1.000	0.285	1.000	0.293	1.000	0.086	1.000	0.063	0.933	0.927	0.565	0.929	0.367	0.926	0.365	0.939	0.055	0.936	0.049
			0.2	1.000	1.000	1.000	1.000	0.730	1.000	0.720	1.000	0.152	1.000	0.062	0.991	0.989	0.789	0.988	0.603	0.987	0.639	0.987	0.070	0.992	0.038
			0.3	1.000	1.000	1.000	1.000	0.953	1.000	0.969	1.000	0.237	1.000	0.074	0.998	0.996	0.874	0.997	0.757	0.997	0.755	0.998	0.087	0.997	0.029
			0.4	1.000	1.000	1.000	1.000	0.998	1.000	0.998	1.000	0.276	1.000	0.079	0.999	0.999	0.909	0.998	0.836	0.999	0.837	0.998	0.095	0.999	0.023
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.336	1.000	0.088	1.000	0.999	0.934	1.000	0.886	1.000	0.885	1.000	0.117	0.999	0.019
		$[1, t]'$	0.1	0.999	0.999	0.995	0.999	0.346	0.999	0.332	0.999	0.092	0.999	0.064	0.950	0.944	0.582	0.936	0.474	0.938	0.464	0.940	0.078	0.953	0.062
			0.2	1.000	1.000	1.000	1.000	0.778	1.000	0.782	1.000	0.164	1.000	0.067	0.997	0.996	0.833	0.997	0.770	0.996	0.773	0.997	0.113	0.997	0.045
			0.3	1.000	1.000	1.000	1.000	0.961	1.000	0.959	1.000	0.263	1.000	0.071	1.000	0.999	0.919	1.000	0.912	1.000	0.890	0.999	0.147	1.000	0.032
			0.4	1.000	1.000	1.000	1.000	0.996	1.000	0.996	1.000	0.322	1.000	0.087	1.000	1.000	0.962	1.000	0.944	1.000	0.950	1.000	0.173	1.000	0.026
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.401	1.000	0.103	1.000	1.000	0.975	1.000	0.973	1.000	0.975	1.000	0.185	1.000	0.020
	0.9	0	0.1	1.000	1.000	1.000	1.000	0.916	1.000	0.906	1.000	0.441	1.000	0.181	0.841	0.832	0.493	0.820	0.333	0.828	0.337	0.824	0.241	0.820	0.121
			0.2	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.714	1.000	0.280	0.950	0.941	0.670	0.940	0.554	0.940	0.547	0.935	0.340	0.938	0.161
			0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.885	1.000	0.293	0.976	0.968	0.749	0.969	0.661	0.972	0.672	0.970	0.436	0.968	0.169
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.951	1.000	0.353	0.988	0.981	0.797	0.982	0.745	0.982	0.739	0.980	0.477	0.982	0.202
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	1.000	0.414	0.994	0.988	0.838	0.986	0.773	0.990	0.778	0.988	0.554	0.991	0.181
		1	0.1	1.000	1.000	1.000	1.000	0.293	1.000	0.266	1.000	0.087	1.000	0.065	0.929	0.924	0.552	0.932	0.358	0.927	0.347	0.934	0.053	0.938	0.052
			0.2	1.000	1.000	1.000	1.000	0.702	1.000	0.725	1.000	0.158	1.000	0.065	0.989	0.987	0.796	0.990	0.614	0.987	0.632	0.988	0.071	0.990	0.034
			0.3	1.000	1.000	1.000	1.000	0.947	1.000	0.959	1.000	0.226	1.000	0.070	0.998	0.997	0.869	0.997	0.744	0.997	0.739	0.997	0.078	0.998	0.027
			0.4	1.000	1.000	1.000	1.000	0.995	1.000	0.995	1.000	0.267	1.000	0.080	0.999	0.999	0.908	0.999	0.830	0.998	0.843	1.000	0.098	0.999	0.022
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.328	1.000	0.089	1.000	1.000	0.937	1.000	0.874	0.999	0.872	1.000	0.101	1.000	0.016
		$[1, t]'$	0.1	0.999	0.999	0.994	0.999	0.348	0.998	0.349	0.999	0.093	0.999	0.065	0.938	0.934	0.570	0.936	0.464	0.930	0.468	0.942	0.075	0.953	0.061
			0.2	1.000	1.000	1.000	1.000	0.775	1.000	0.781	1.000	0.164	1.000	0.063	0.996	0.995	0.838	0.995	0.750	0.997	0.771	0.997	0.113	0.997	0.047
			0.3	1.000	1.000	1.000	1.000	0.958	1.000	0.944	1.000	0.263	1.000	0.077	1.000	0.999	0.920	0.999	0.889	1.000	0.883	1.000	0.143	1.000	0.031
			0.4	1.000	1.000	1.000	1.000	0.996	1.000	0.995	1.000	0.328	1.000	0.084	1.000	1.000	0.957	1.000	0.939	1.000	0.945	1.000	0.152	1.000	0.027
			0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.399	1.000	0.098	1.000	1.000	0.976	1.000	0.971	1.000	0.971	1.000	0.180	1.000	0.020

Note: Simulations are based on DGP (b). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, ρ_c demonstrates the AR(1) coefficient of the process $z_{2,t}$.

Table 10: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 0$ vs Fractional Cointegration with Sample Size=128

T	θ	δ_t	b	$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
128	-0.9	0	0.2	0.288	0.243	0.616	0.956	0.421	0.960	0.428	0.948	0.148	0.930	0.111	0.783	0.603	0.374	0.579	0.153	0.581	0.157	0.563	0.094	0.495	0.098
			0.4	0.737	0.651	0.977	1.000	0.857	1.000	0.834	1.000	0.309	1.000	0.222	0.958	0.872	0.692	0.850	0.404	0.850	0.413	0.829	0.213	0.793	0.184
			0.6	0.976	0.937	1.000	1.000	0.992	1.000	0.990	1.000	0.409	1.000	0.335	0.999	0.984	0.927	0.975	0.731	0.976	0.721	0.972	0.456	0.954	0.383
			0.8	0.999	0.992	1.000	1.000	1.000	1.000	0.999	1.000	0.521	1.000	0.385	1.000	1.000	0.997	0.999	0.938	1.000	0.935	0.999	0.740	0.999	0.644
			1	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	0.613	1.000	0.428	1.000	1.000	1.000	1.000	0.995	1.000	0.995	1.000	0.930	1.000	0.898
		1	0.2	0.276	0.223	0.812	0.998	0.238	0.998	0.230	0.998	0.060	0.997	0.034	0.957	0.850	0.559	0.839	0.268	0.845	0.265	0.847	0.038	0.851	0.021
			0.4	0.586	0.516	0.993	1.000	0.414	1.000	0.376	1.000	0.095	1.000	0.042	1.000	0.985	0.883	0.979	0.576	0.981	0.560	0.979	0.067	0.978	0.021
			0.6	0.765	0.688	1.000	1.000	0.509	1.000	0.549	1.000	0.120	1.000	0.054	1.000	1.000	0.993	0.999	0.839	0.999	0.837	0.998	0.152	0.998	0.024
			0.8	0.896	0.807	1.000	1.000	0.609	1.000	0.615	1.000	0.144	1.000	0.060	1.000	1.000	1.000	1.000	0.949	1.000	0.947	1.000	0.315	1.000	0.049
			1	0.961	0.853	1.000	1.000	0.655	1.000	0.641	1.000	0.163	1.000	0.068	1.000	1.000	1.000	1.000	0.977	1.000	0.981	1.000	0.512	1.000	0.092
		$[1, t]'$	0.2	0.226	0.190	0.755	0.999	0.359	0.999	0.352	0.998	0.073	0.999	0.038	0.996	0.952	0.723	0.942	0.430	0.946	0.441	0.945	0.048	0.928	0.023
			0.4	0.448	0.384	0.956	1.000	0.511	1.000	0.521	1.000	0.114	1.000	0.046	1.000	0.999	0.965	0.997	0.713	0.996	0.710	0.996	0.089	0.990	0.020
			0.6	0.614	0.540	0.994	1.000	0.670	1.000	0.652	1.000	0.145	1.000	0.053	1.000	1.000	0.999	0.999	0.872	0.999	0.863	0.999	0.167	0.999	0.032
			0.8	0.730	0.634	1.000	1.000	0.715	1.000	0.724	1.000	0.175	1.000	0.064	1.000	1.000	1.000	1.000	0.920	1.000	0.922	1.000	0.314	1.000	0.046
			1	0.789	0.681	1.000	1.000	0.775	1.000	0.767	1.000	0.190	1.000	0.072	1.000	1.000	1.000	1.000	0.957	1.000	0.955	1.000	0.430	1.000	0.090
	-0.5	0	0.2	0.255	0.159	0.114	0.202	0.049	0.209	0.052	0.185	0.039	0.142	0.046	0.167	0.109	0.043	0.111	0.029	0.107	0.025	0.091	0.041	0.073	0.051
			0.4	0.763	0.495	0.411	0.553	0.112	0.568	0.109	0.525	0.052	0.461	0.056	0.409	0.259	0.100	0.240	0.039	0.246	0.042	0.220	0.049	0.176	0.059
			0.6	0.996	0.916	0.876	0.939	0.306	0.940	0.311	0.923	0.116	0.885	0.101	0.755	0.537	0.262	0.503	0.090	0.500	0.092	0.480	0.061	0.414	0.084
			0.8	1.000	1.000	0.999	1.000	0.669	1.000	0.689	1.000	0.265	0.999	0.200	0.963	0.841	0.612	0.813	0.279	0.815	0.275	0.789	0.126	0.731	0.141
			1	1.000	1.000	1.000	1.000	0.961	1.000	0.959	1.000	0.398	1.000	0.313	1.000	0.988	0.918	0.974	0.652	0.976	0.648	0.968	0.324	0.946	0.332
		1	0.2	0.325	0.202	0.142	0.321	0.029	0.330	0.032	0.291	0.033	0.263	0.032	0.267	0.179	0.047	0.171	0.024	0.184	0.025	0.185	0.032	0.196	0.034
			0.4	0.891	0.619	0.513	0.749	0.057	0.757	0.056	0.702	0.028	0.677	0.025	0.618	0.405	0.124	0.405	0.054	0.426	0.057	0.424	0.023	0.432	0.027
			0.6	1.000	0.962	0.924	0.986	0.137	0.985	0.133	0.976	0.040	0.971	0.026	0.926	0.756	0.373	0.741	0.143	0.760	0.137	0.748	0.025	0.756	0.025
			0.8	1.000	1.000	0.999	1.000	0.268	1.000	0.289	1.000	0.063	1.000	0.033	0.998	0.964	0.759	0.955	0.338	0.950	0.345	0.954	0.031	0.947	0.015
			1	1.000	1.000	1.000	1.000	0.433	1.000	0.452	1.000	0.098	1.000	0.040	1.000	0.999	0.969	0.996	0.667	0.996	0.674	0.996	0.060	0.994	0.014
		$[1, t]'$	0.2	0.325	0.197	0.117	0.366	0.043	0.383	0.048	0.346	0.033	0.366	0.029	0.365	0.224	0.047	0.230	0.060	0.238	0.062	0.247	0.027	0.232	0.032
			0.4	0.858	0.568	0.406	0.756	0.087	0.768	0.091	0.736	0.031	0.747	0.025	0.739	0.491	0.147	0.492	0.116	0.511	0.118	0.511	0.020	0.498	0.026
			0.6	0.999	0.921	0.830	0.977	0.198	0.981	0.195	0.970	0.047	0.974	0.027	0.975	0.825	0.421	0.813	0.238	0.829	0.245	0.817	0.027	0.791	0.020
			0.8	1.000	0.998	0.988	1.000	0.393	1.000	0.372	1.000	0.070	1.000	0.032	1.000	0.983	0.817	0.975	0.484	0.976	0.486	0.972	0.042	0.958	0.016
			1	1.000	1.000	1.000	1.000	0.535	1.000	0.532	1.000	0.103	1.000	0.040	1.000	1.000	0.987	0.998	0.739	0.998	0.747	0.998	0.085	0.996	0.017
	0	0	0.2	0.139	0.116	0.112	0.104	0.053	0.108	0.052	0.086	0.052	0.067	0.052	0.082	0.075	0.043	0.076	0.044	0.077	0.041	0.061	0.047	0.042	0.049
			0.4	0.429	0.305	0.296	0.274	0.072	0.291	0.074	0.249	0.058	0.202	0.052	0.187	0.154	0.065	0.150	0.041	0.145	0.043	0.130	0.052	0.095	0.055
			0.6	0.882	0.689	0.680	0.658	0.152	0.659	0.140	0.629	0.079	0.564	0.073	0.418	0.321	0.148	0.323	0.062	0.306	0.060	0.292	0.058	0.232	0.068
			0.8	1.000	0.975	0.973	0.969	0.334	0.971	0.329	0.958	0.150	0.933	0.131	0.755	0.606	0.330	0.582	0.131	0.588	0.141	0.571	0.091	0.501	0.106
			1	1.000	1.000	1.000	1.000	0.656	1.000	0.636	1.000	0.267	1.000	0.211	0.967	0.888	0.674	0.874	0.337	0.867	0.353	0.852	0.172	0.796	0.167
		1	0.2	0.180	0.140	0.115	0.139	0.035	0.140	0.034	0.114	0.050	0.108	0.049	0.114	0.103	0.045	0.103	0.037	0.107	0.038	0.116	0.049	0.121	0.049
			0.4	0.555	0.391	0.344	0.392	0.039	0.386	0.042	0.337	0.046	0.335	0.042	0.278	0.226	0.068	0.240	0.043	0.253	0.043	0.257	0.041	0.264	0.045
			0.6	0.946	0.776	0.732	0.775	0.067	0.781	0.067	0.738	0.043	0.722	0.037	0.601	0.466	0.167	0.489	0.076	0.498	0.076	0.513	0.035	0.519	0.038
			0.8	1.000	0.982	0.973	0.983	0.125	0.982	0.128	0.976	0.053	0.971	0.034	0.908	0.790	0.409	0.791	0.168	0.805	0.157	0.800	0.034	0.802	0.032
			1	1.000	1.000	1.000	1.000	0.241	1.000	0.233	1.000	0.064	1.000	0.039	0.996	0.965	0.757	0.960	0.362	0.964	0.354	0.966	0.046	0.960	0.023
		$[1, t]'$	0.2	0.172	0.133	0.083	0.143	0.043	0.143	0.041	0.131	0.052	0.131	0.050	0.119	0.101	0.035	0.122	0.058	0.119	0.058	0.132	0.052	0.132	0.049
			0.4	0.534	0.353	0.257	0.356	0.056	0.373	0.062	0.337	0.045	0.363	0.043	0.307	0.240	0.063	0.260	0.075	0.268	0.086	0.278	0.045	0.274	0.045
			0.6	0.918	0.696	0.585	0.710	0.096	0.734	0.093	0.700	0.045	0.726	0.038	0.662	0.517	0.168	0.523	0.134	0.542	0.133	0.541	0.035	0.525	0.037
			0.8	1.000	0.954	0.904	0.953	0.180	0.957	0.187	0.952	0.056	0.953	0.036	0.942	0.821	0.403	0.830	0.256	0.828	0.264	0.825	0.039	0.792	0.031
			1	1.000	0.999	0.996	0.999	0.315	0.999	0.321	0.999	0.078	0.999	0.035	0.999	0.980	0.766	0.971	0.478	0.973	0.491	0.964	0.048	0.943	0.023
	0.5	0	0.2	0.115	0.112	0.107	0.091	0.065	0.096	0.055	0.078	0.056	0.061	0.057	0.073	0.073	0.049	0.068	0.046	0.069	0.048	0.059	0.052	0.040	0.054
			0.4	0.312	0.266	0.257	0.245	0.076	0.245	0.082	0.217	0.059	0.172	0.055	0.141	0.133	0.068	0.133	0.055	0.137	0.052	0.120	0.055	0.086	0.057
			0.6	0.719	0.606	0.593	0.573	0.141	0.593	0.151	0.543	0.077	0.478	0.076	0.325	0.288	0.129	0.278	0.071	0.283	0.069	0.254	0.064	0.212	0.062
			0.8	0.986	0.944	0.941	0.931	0.312	0.935	0.327	0.916	0.137	0.880	0.116	0.629	0.555	0.294	0.541	0.134	0.527	0.143	0.525	0.085	0.447	0.088
			1.000	1.000	1.000	1.000	0.999	0.575	1.000	0.587	1.000	0.247	0.997	0.189	0.904	0.846	0.598	0.830	0.312	0.827	0.330	0.821	0.152	0.750	0.156
		1	0.2	0.138	0.127	0.112	0.126	0.042	0.130	0.043	0.099	0.056	0.088	0.052	0.089	0.089	0.050	0.101	0.046	0.101	0.045	0.107	0.054	0.109	0.051
			0.4	0.386	0.327	0.299	0.335	0.050	0.338	0.052	0.301	0.057	0.273	0.053	0.214	0.201	0.081	0.207	0.052	0.220	0.049	0.240	0.048	0.233	0.050
			0.6	0.795	0.677	0.641	0.693	0.076	0.689	0.074	0.646	0.062	0.635	0.051	0.473	0.424	0.150	0.443	0.082	0.452	0.078	0.466	0.047	0.468	0.049
			0.8	0.993	0.955	0.944	0.950	0.131	0.954	0.132	0.940	0.064	0.935	0.048	0.794	0.723	0.356	0.733	0.149	0.751	0.156	0.751	0.047	0.758	0.047
			1	1.000	0.999	0.999	0.999	0.221	1.000	0.218	0.999	0.082	0.999	0.051	0.974	0.942	0.667	0.941	0.326	0.943	0.334	0.943	0.063	0.936	0.041
		$[1, t]'$	0.2	0.136	0.126	0.084	0.110	0.051	0.123	0.051	0.108	0.059	0.109	0.052	0.097	0.098	0.044	0.105	0.064	0.111	0.062	0.119	0.054	0.127	0.052
			0.4	0.368	0.306	0.225	0.287	0.067	0.300	0.066	0.274	0.056	0.299	0.054	0.214	0.206	0.070	0.236	0.090	0.242	0.086	0.248	0.050	0.244	0.052
			0.6	0.762	0.628	0.527	0.599	0.100	0.617	0.095	0.590	0.066	0.625	0.050	0.477	0.423	0.141	0.473	0.145	0.477	0.140	0.487	0.051	0.466	0.050
			0.8	0.979	0.911	0.849	0.895	0.170	0.903	0.166	0.895	0.072	0.912	0.049	0.807	0.730	0.333	0.760	0.260	0.778	0.256	0.776	0.053	0.723	0.045
			1	1	0.996	0.988	0.994	0.286	0.994	0.296	0.993	0.088	0.994	0.048	0.983	0.951	0.677	0.951	0.465	0.951	0.448	0.946	0.074	0.910	0.041
	0.9	0	0.2	0.110	0.106	0.104	0.092	0.058	0.085	0.059	0.080	0.058	0.059	0.055	0.070	0.071	0.048	0.071	0.051	0.072	0.053	0.061	0.053	0.040	0.050
			0.4	0.290	0.256	0.253	0.230	0.082	0.241	0.078	0.214	0.064	0.167	0.054	0.139	0.133	0.072	0.133	0.056	0.133	0.055	0.124	0.055	0.090	0.056
			0.6	0.684	0.594	0.590	0.568	0.142	0.571	0.141	0.540	0.082	0.471	0.075	0.308	0.282	0.127	0.279	0.080	0.280	0.080	0.264	0.062	0.209	0.064
			0.8	0.977	0.932	0.929	0.920	0.320	0.932	0.313	0.905	0.144	0.873	0.110	0.595	0.540	0.284	0.524	0.145	0.533	0.136	0.506	0.079	0.436	0.097
			1	1.000	1.000	1.000	0.999	0.600	1.000	0.583	0.999	0.249	0.996	0.186	0.881	0.835	0.602	0.830	0.322	0.827	0.317	0.805	0.154	0.751	0.150
		1	0.2	0.145	0.137	0.115	0.120	0.042	0.116	0.046	0.091	0.058	0.086	0.052	0.086	0.088	0.049	0.095	0.045	0.105	0.043	0.105	0.053	0.113	0.052
			0.4	0.384	0.341	0.304	0.318	0.050	0.333	0.050	0.285	0.057	0.275	0.053	0.205	0.197	0.071	0.211	0.052	0.223	0.050	0.228	0.051	0.236	0.052
			0.6	0.784	0.692	0.650	0.673	0.074	0.672	0.076	0.629	0.060	0.626	0.055	0.448	0.412	0.148	0.438	0.080	0.447	0.085	0.472	0.049	0.472	0.051
			0.8	0.987	0.958	0.940	0.949	0.126	0.948	0.121	0.935	0.073	0.927	0.052	0.766	0.709	0.342	0.730	0.155	0.737	0.158	0.750	0.050	0.748	0.048
			1	1.000	1.000	0.999	0.999	0.219	1.000	0.221	0.999	0.090	0.998	0.054	0.961	0.931	0.672	0.939	0.321	0.938	0.336	0.939	0.070	0.938	0.045
		$[1, t]'$	0.2	0.144	0.134	0.093	0.106	0.052	0.112	0.049	0.102	0.062	0.108	0.054	0.095	0.099	0.039	0.102	0.062	0.109	0.060	0.117	0.055	0.118	0.052
			0.4	0.354	0.314	0.239	0.285	0.071	0.293	0.069	0.271	0.061	0.286	0.054	0.217	0.212	0.067	0.226	0.090	0.236	0.091	0.245	0.053	0.234	0.053
			0.6	0.724	0.633	0.532	0.575	0.108	0.590	0.100	0.577	0.065	0.600	0.057	0.454	0.424	0.144	0.460	0.141	0.476	0.142	0.479	0.051	0.459	0.052
			0.8	0.967	0.907	0.850	0.888	0.174	0.893	0.177	0.884	0.080	0.903	0.053	0.787	0.724	0.332	0.764	0.261	0.769	0.260	0.773	0.057	0.721	0.053
			1	1.000	0.995	0.987	0.992	0.290	0.994	0.299	0.992	0.095	0.992	0.057	0.971	0.942	0.665	0.955	0.476	0.958	0.463	0.950	0.068	0.908	0.049

Note: Simulations are based on DGP (d). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 0$ and $p = 2$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 0$ and $p = 2$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, $b - 1$ demonstrates the fractional integration order of $z_{2,t}$.

Table 11: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 0$ vs Fractional Cointegration with Sample Size=1024

				$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}	Λ^L	Λ^{L*}
T	θ	δ_t	b		Haar		D_2		S_2		S_4		S_8			Haar		D_2		S_2		S_4		S_8	
1024	-0.9	0	0.2	0.750	0.631	0.621	0.928	0.785	0.927	0.783	0.914	0.306	0.915	0.084	0.616	0.418	0.111	0.401	0.108	0.419	0.118	0.406	0.177	0.389	0.068
			0.4	1.000	0.998	0.998	1.000	0.999	1.000	1.000	1.000	0.902	1.000	0.435	0.956	0.861	0.554	0.849	0.516	0.854	0.523	0.851	0.497	0.839	0.161
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.967	1.000	0.996	0.943	0.995	0.938	0.995	0.932	0.994	0.881	0.995	0.552
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.953
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		1	0.2	0.886	0.780	0.833	0.995	0.779	0.997	0.775	0.996	0.140	0.995	0.037	0.816	0.611	0.174	0.618	0.246	0.613	0.229	0.629	0.032	0.626	0.021
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.483	1.000	0.115	0.999	0.974	0.754	0.973	0.782	0.974	0.786	0.974	0.106	0.971	0.022
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.824	1.000	0.230	1.000	1.000	0.996	1.000	0.998	1.000	0.998	1.000	0.380	1.000	0.067
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.915	1.000	0.301	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.872	1.000	0.294
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.972	1.000	0.329	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.993	1.000	0.775
		$[1, t]'$	0.2	0.939	0.838	0.943	1.000	0.938	1.000	0.935	1.000	0.202	1.000	0.049	0.956	0.807	0.293	0.817	0.478	0.802	0.482	0.807	0.051	0.822	0.023
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.601	1.000	0.148	1.000	0.999	0.906	0.999	0.959	0.998	0.961	0.999	0.183	0.999	0.038
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.853	1.000	0.302	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.550	1.000	0.105
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.914	1.000	0.360	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.959	1.000	0.397
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.943	1.000	0.399	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.766
	-0.5	0	0.2	0.281	0.192	0.140	0.223	0.090	0.221	0.090	0.215	0.068	0.212	0.041	0.119	0.099	0.025	0.095	0.018	0.098	0.019	0.100	0.067	0.097	0.057
			0.4	0.916	0.743	0.677	0.770	0.458	0.775	0.454	0.763	0.140	0.754	0.041	0.393	0.281	0.057	0.263	0.037	0.272	0.044	0.276	0.108	0.275	0.057
			0.6	1.000	1.000	0.999	0.999	0.989	1.000	0.991	0.999	0.456	0.999	0.116	0.862	0.702	0.309	0.686	0.262	0.692	0.260	0.692	0.280	0.673	0.114
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994	1.000	0.475	0.999	0.981	0.827	0.975	0.812	0.977	0.799	0.975	0.759	0.974	0.280
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.944	1.000	1.000	0.999	1.000	0.997	1.000	0.997	1.000	0.990	1.000	0.831
		1	0.2	0.442	0.297	0.179	0.319	0.029	0.340	0.033	0.330	0.024	0.325	0.034	0.157	0.125	0.022	0.127	0.020	0.129	0.021	0.141	0.026	0.141	0.042
			0.4	0.993	0.916	0.841	0.920	0.262	0.933	0.248	0.927	0.035	0.924	0.017	0.586	0.428	0.066	0.422	0.089	0.431	0.093	0.446	0.020	0.444	0.024
			0.6	1.000	1.000	1.000	1.000	0.946	1.000	0.946	1.000	0.111	1.000	0.023	0.975	0.893	0.430	0.885	0.469	0.886	0.451	0.891	0.039	0.892	0.015
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.354	1.000	0.071	1.000	0.999	0.947	0.999	0.942	0.999	0.951	0.999	0.138	0.999	0.015
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.699	1.000	0.181	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.475	1.000	0.042
		$[1, t]'$	0.2	0.512	0.334	0.147	0.389	0.060	0.387	0.057	0.391	0.027	0.376	0.033	0.210	0.158	0.018	0.164	0.042	0.156	0.044	0.161	0.028	0.175	0.043
			0.4	0.998	0.945	0.836	0.957	0.389	0.957	0.378	0.956	0.043	0.954	0.018	0.712	0.521	0.073	0.533	0.190	0.500	0.178	0.532	0.026	0.543	0.028
			0.6	1.000	1.000	1.000	1.000	0.977	1.000	0.981	1.000	0.121	1.000	0.023	0.997	0.951	0.485	0.951	0.697	0.941	0.677	0.952	0.059	0.958	0.016
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.440	1.000	0.085	1.000	1.000	0.978	1.000	0.996	1.000	0.994	1.000	0.185	1.000	0.022
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.746	1.000	0.205	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.579	1.000	0.059
	0	0	0.2	0.156	0.138	0.123	0.159	0.078	0.159	0.075	0.158	0.078	0.143	0.051	0.095	0.092	0.044	0.080	0.039	0.088	0.034	0.084	0.066	0.082	0.058
			0.4	0.662	0.564	0.538	0.583	0.262	0.583	0.259	0.583	0.127	0.565	0.058	0.233	0.210	0.055	0.208	0.044	0.217	0.044	0.215	0.098	0.206	0.059
			0.6	0.997	0.986	0.983	0.987	0.858	0.988	0.875	0.985	0.349	0.987	0.096	0.619	0.540	0.179	0.535	0.156	0.537	0.157	0.535	0.201	0.536	0.099
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.900	1.000	0.283	0.962	0.918	0.617	0.910	0.584	0.914	0.598	0.910	0.491	0.911	0.180
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.602	1.000	0.999	0.970	1.000	0.972	1.000	0.964	0.999	0.937	1.000	0.504
		1	0.2	0.239	0.211	0.171	0.207	0.025	0.234	0.029	0.220	0.043	0.218	0.053	0.117	0.113	0.034	0.110	0.024	0.111	0.027	0.125	0.043	0.125	0.051
			0.4	0.834	0.731	0.677	0.739	0.111	0.763	0.102	0.766	0.044	0.763	0.042	0.356	0.318	0.063	0.315	0.065	0.327	0.058	0.342	0.030	0.345	0.043
			0.6	1.000	0.999	0.998	0.998	0.594	0.999	0.593	1.000	0.096	0.999	0.035	0.819	0.745	0.256	0.738	0.262	0.757	0.264	0.767	0.034	0.770	0.031
			0.8	1.000	1.000	1.000	1.000	0.999	1.000	0.999	1.000	0.239	1.000	0.052	0.996	0.985	0.752	0.984	0.776	0.986	0.757	0.988	0.092	0.988	0.026
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.513	1.000	0.116	1.000	1.000	0.995	1.000	0.995	1.000	0.997	1.000	0.295	1.000	0.036
		$[1, t]'$	0.2	0.263	0.225	0.140	0.237	0.043	0.234	0.044	0.238	0.045	0.227	0.057	0.133	0.126	0.036	0.133	0.043	0.126	0.044	0.127	0.042	0.136	0.053
			0.4	0.879	0.766	0.654	0.782	0.158	0.782	0.171	0.798	0.054	0.787	0.042	0.423	0.370	0.067	0.385	0.111	0.364	0.112	0.377	0.034	0.411	0.047
			0.6	1.000	0.999	0.997	0.999	0.701	1.000	0.696	0.999	0.100	1.000	0.035	0.899	0.827	0.282	0.828	0.429	0.824	0.436	0.828	0.049	0.848	0.035
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.276	1.000	0.058	1.000	0.997	0.830	0.998	0.918	0.997	0.914	0.998	0.143	0.998	0.030
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.581	1.000	0.129	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.387	1.000	0.046
	0.5	0	0.2	0.148	0.139	0.130	0.150	0.077	0.156	0.079	0.142	0.078	0.146	0.057	0.086	0.085	0.041	0.079	0.042	0.086	0.042	0.086	0.068	0.090	0.057
			0.4	0.587	0.542	0.525	0.548	0.219	0.550	0.237	0.545	0.121	0.526	0.063	0.219	0.211	0.060	0.193	0.052	0.207	0.053	0.211	0.098	0.198	0.066
			0.6	0.990	0.976	0.974	0.982	0.792	0.980	0.795	0.980	0.327	0.978	0.113	0.547	0.513	0.165	0.495	0.139	0.526	0.144	0.518	0.181	0.511	0.088
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.841	1.000	0.282	0.922	0.891	0.557	0.888	0.506	0.893	0.531	0.888	0.449	0.896	0.202
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.588	0.999	0.998	0.951	0.999	0.948	0.998	0.946	0.998	0.897	0.999	0.450
		1	0.2	0.201	0.191	0.173	0.200	0.028	0.205	0.030	0.215	0.045	0.203	0.054	0.107	0.105	0.044	0.100	0.030	0.110	0.029	0.122	0.046	0.120	0.052
			0.4	0.753	0.708	0.684	0.707	0.095	0.732	0.087	0.733	0.051	0.734	0.050	0.321	0.308	0.067	0.309	0.065	0.313	0.060	0.332	0.037	0.339	0.049
			0.6	0.999	0.997	0.996	0.997	0.493	0.998	0.492	0.998	0.102	0.998	0.045	0.754	0.719	0.236	0.715	0.234	0.727	0.222	0.737	0.040	0.739	0.041
			0.8	1.000	1.000	1.000	1.000	0.990	1.000	0.990	1.000	0.260	1.000	0.067	0.988	0.980	0.717	0.978	0.686	0.979	0.696	0.981	0.094	0.982	0.035
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.565	1.000	0.134	1.000	1.000	0.990	1.000	0.988	1.000	0.985	1.000	0.294	1.000	0.041
		$[1, t]'$	0.2	0.242	0.228	0.153	0.218	0.043	0.217	0.043	0.218	0.054	0.211	0.057	0.127	0.125	0.037	0.125	0.048	0.115	0.044	0.123	0.048	0.138	0.054
			0.4	0.804	0.753	0.649	0.738	0.145	0.737	0.138	0.749	0.062	0.740	0.051	0.377	0.355	0.075	0.362	0.102	0.353	0.105	0.374	0.036	0.390	0.050
			0.6	1.000	0.998	0.996	0.999	0.588	0.999	0.603	0.999	0.121	0.999	0.053	0.824	0.783	0.260	0.786	0.348	0.788	0.353	0.800	0.057	0.822	0.044
			0.8	1.000	1.000	1.000	1.000	0.996	1.000	0.996	1.000	0.310	1.000	0.077	0.998	0.994	0.784	0.995	0.848	0.995	0.859	0.995	0.148	0.996	0.036
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.609	1.000	0.156	1.000	1.000	0.998	1.000	1.000	1.000	0.999	1.000	0.377	1.000	0.057
	0.9	0	0.2	0.148	0.144	0.136	0.159	0.079	0.158	0.077	0.142	0.073	0.138	0.063	0.093	0.092	0.048	0.083	0.043	0.089	0.047	0.088	0.070	0.086	0.056
			0.4	0.567	0.531	0.516	0.546	0.231	0.543	0.241	0.530	0.121	0.529	0.066	0.208	0.201	0.059	0.189	0.050	0.207	0.049	0.209	0.085	0.195	0.066
			0.6	0.988	0.979	0.976	0.978	0.768	0.980	0.762	0.977	0.342	0.977	0.108	0.531	0.502	0.168	0.503	0.131	0.520	0.139	0.516	0.179	0.517	0.087
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.828	1.000	0.297	0.918	0.891	0.574	0.885	0.491	0.891	0.492	0.895	0.418	0.890	0.184
			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.596	1.000	0.999	0.951	0.999	0.937	0.999	0.941	0.999	0.871	0.999	0.438
		1	0.2	0.202	0.193	0.162	0.194	0.031	0.217	0.034	0.202	0.050	0.199	0.057	0.107	0.106	0.049	0.110	0.031	0.109	0.031	0.119	0.047	0.122	0.054
			0.4	0.744	0.702	0.658	0.700	0.083	0.730	0.096	0.713	0.055	0.724	0.051	0.321	0.309	0.071	0.297	0.063	0.314	0.062	0.339	0.035	0.328	0.050
			0.6	0.999	0.998	0.996	0.998	0.437	0.997	0.454	0.997	0.098	0.998	0.048	0.739	0.709	0.237	0.711	0.219	0.718	0.228	0.750	0.041	0.738	0.040
			0.8	1.000	1.000	1.000	1.000	0.983	1.000	0.983	1.000	0.273	1.000	0.074	0.984	0.975	0.714	0.977	0.663	0.982	0.638	0.982	0.087	0.984	0.032
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.531	1.000	0.138	1.000	1.000	0.988	1.000	0.981	1.000	0.982	1.000	0.269	1.000	0.044
		$[1, t]'$	0.2	0.227	0.215	0.149	0.217	0.043	0.212	0.046	0.223	0.055	0.208	0.057	0.123	0.121	0.040	0.126	0.047	0.120	0.046	0.129	0.048	0.141	0.055
			0.4	0.774	0.726	0.637	0.733	0.135	0.732	0.144	0.740	0.063	0.737	0.053	0.371	0.354	0.077	0.367	0.105	0.351	0.109	0.366	0.041	0.382	0.051
			0.6	0.999	0.998	0.995	0.998	0.563	0.999	0.552	0.999	0.119	0.998	0.052	0.815	0.779	0.271	0.796	0.365	0.787	0.342	0.795	0.058	0.818	0.042
			0.8	1.000	1.000	1.000	1.000	0.991	1.000	0.990	1.000	0.308	1.000	0.079	0.997	0.994	0.776	0.995	0.822	0.995	0.823	0.996	0.139	0.996	0.042
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.602	1.000	0.152	1.000	1.000	0.998	1.000	0.999	1.000	0.999	1.000	0.364	1.000	0.051

Note: Simulations are based on DGP (d). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 0$ and $p = 2$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 0$ and $p = 2$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are $Haar$ and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, $b - 1$ demonstrates the fractional integration order of $z_{2,t}$.

Table 12: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 1$ vs Fractional Cointegration with Sample Size=128

T	θ	δ_t	b	$d_1=0.1$ Λ	Haar Λ^L	Haar Λ^{L*}	D_2 Λ^L	D_2 Λ^{L*}	S_2 Λ^L	S_2 Λ^{L*}	S_4 Λ^L	S_4 Λ^{L*}	S_8 Λ^L	S_8 Λ^{L*}	$d_1=1$ Λ	Haar Λ^L	Haar Λ^{L*}	D_2 Λ^L	D_2 Λ^{L*}	S_2 Λ^L	S_2 Λ^{L*}	S_4 Λ^L	S_4 Λ^{L*}	S_8 Λ^L	S_8 Λ^{L*}
128	-0.9	0	0.2	0.212	0.185	0.423	0.958	0.301	0.956	0.304	0.943	0.103	0.911	0.071	0.739	0.492	0.201	0.445	0.048	0.465	0.053	0.408	0.020	0.343	0.027
			0.4	0.372	0.384	0.720	1.000	0.512	1.000	0.549	1.000	0.206	0.997	0.151	0.913	0.735	0.420	0.689	0.124	0.701	0.120	0.647	0.042	0.563	0.040
			0.6	0.394	0.445	0.772	1.000	0.602	1.000	0.573	1.000	0.237	0.999	0.191	0.963	0.856	0.598	0.814	0.219	0.823	0.241	0.795	0.088	0.722	0.085
			0.8	0.400	0.454	0.781	1.000	0.598	1.000	0.586	1.000	0.264	0.999	0.198	0.968	0.879	0.658	0.852	0.274	0.858	0.308	0.829	0.131	0.770	0.131
			1	0.398	0.451	0.785	1.000	0.608	1.000	0.585	1.000	0.236	1.000	0.196	0.967	0.890	0.673	0.858	0.326	0.867	0.338	0.846	0.162	0.786	0.145
		1	0.2	0.186	0.171	0.550	0.997	0.120	0.996	0.119	0.994	0.032	0.994	0.020	0.935	0.773	0.354	0.773	0.098	0.762	0.101	0.767	0.010	0.770	0.009
			0.4	0.262	0.286	0.722	1.000	0.178	1.000	0.184	1.000	0.049	1.000	0.025	0.992	0.923	0.610	0.919	0.190	0.919	0.191	0.919	0.011	0.911	0.008
			0.6	0.261	0.299	0.734	1.000	0.203	1.000	0.197	1.000	0.055	0.999	0.029	0.995	0.961	0.714	0.954	0.283	0.949	0.291	0.952	0.021	0.943	0.011
			0.8	0.266	0.309	0.752	1.000	0.196	1.000	0.203	1.000	0.054	0.999	0.031	0.996	0.962	0.735	0.959	0.326	0.958	0.312	0.959	0.031	0.952	0.018
			1	0.270	0.306	0.752	1.000	0.186	1.000	0.192	1.000	0.051	1.000	0.030	0.996	0.965	0.762	0.958	0.344	0.958	0.338	0.959	0.037	0.955	0.018
		$[1, t]'$	0.2	0.137	0.132	0.510	0.997	0.188	0.996	0.190	0.995	0.038	0.995	0.019	0.991	0.893	0.452	0.867	0.198	0.879	0.185	0.871	0.008	0.797	0.007
			0.4	0.164	0.185	0.603	0.998	0.242	0.999	0.250	0.998	0.051	0.998	0.024	0.999	0.967	0.666	0.949	0.295	0.949	0.315	0.938	0.015	0.892	0.006
			0.6	0.179	0.209	0.628	0.998	0.263	0.999	0.253	0.999	0.052	0.999	0.029	0.999	0.974	0.737	0.963	0.374	0.961	0.366	0.954	0.023	0.916	0.010
			0.8	0.182	0.211	0.639	0.999	0.258	0.999	0.256	0.998	0.051	0.999	0.027	0.999	0.978	0.752	0.966	0.416	0.967	0.394	0.960	0.033	0.926	0.015
			1	0.178	0.212	0.641	0.999	0.239	0.999	0.247	0.998	0.051	0.999	0.027	0.999	0.973	0.750	0.970	0.433	0.968	0.432	0.964	0.036	0.932	0.017
	-0.5	0	0.2	0.259	0.158	0.101	0.206	0.039	0.199	0.041	0.180	0.038	0.131	0.036	0.149	0.090	0.027	0.084	0.012	0.087	0.012	0.061	0.024	0.042	0.030
			0.4	0.774	0.495	0.365	0.558	0.080	0.538	0.086	0.509	0.043	0.431	0.045	0.360	0.198	0.052	0.184	0.013	0.189	0.015	0.157	0.019	0.116	0.029
			0.6	0.995	0.911	0.831	0.933	0.242	0.928	0.230	0.910	0.085	0.850	0.065	0.696	0.429	0.161	0.390	0.030	0.395	0.029	0.346	0.014	0.271	0.026
			0.8	1.000	0.999	0.996	0.999	0.505	0.999	0.497	0.999	0.182	0.995	0.138	0.912	0.708	0.366	0.646	0.083	0.662	0.089	0.611	0.025	0.514	0.031
			1	1.000	1.000	0.999	1.000	0.604	1.000	0.587	1.000	0.244	0.999	0.192	0.962	0.855	0.572	0.820	0.191	0.813	0.199	0.782	0.059	0.703	0.078
		1	0.2	0.346	0.224	0.123	0.301	0.021	0.292	0.021	0.267	0.022	0.249	0.024	0.252	0.149	0.030	0.160	0.018	0.152	0.016	0.152	0.024	0.172	0.029
			0.4	0.885	0.624	0.438	0.729	0.037	0.714	0.036	0.696	0.020	0.659	0.017	0.592	0.356	0.074	0.365	0.027	0.353	0.027	0.371	0.012	0.384	0.017
			0.6	1.000	0.956	0.868	0.978	0.090	0.973	0.078	0.968	0.022	0.960	0.013	0.906	0.679	0.236	0.674	0.057	0.659	0.057	0.673	0.008	0.675	0.008
			0.8	1.000	0.999	0.989	1.000	0.153	1.000	0.155	0.999	0.036	0.999	0.020	0.988	0.893	0.502	0.882	0.121	0.886	0.134	0.886	0.006	0.872	0.006
			1	1.000	1.000	0.993	1.000	0.192	1.000	0.189	1.000	0.050	1.000	0.029	0.994	0.953	0.685	0.944	0.223	0.943	0.243	0.947	0.013	0.936	0.007
		$[1, t]'$	0.2	0.320	0.199	0.084	0.340	0.029	0.365	0.031	0.334	0.021	0.335	0.021	0.345	0.193	0.030	0.185	0.035	0.189	0.033	0.200	0.020	0.177	0.020
			0.4	0.844	0.551	0.323	0.715	0.052	0.734	0.053	0.715	0.020	0.728	0.016	0.718	0.429	0.089	0.425	0.058	0.428	0.060	0.437	0.009	0.377	0.010
			0.6	0.998	0.886	0.685	0.967	0.115	0.967	0.117	0.957	0.021	0.957	0.014	0.963	0.746	0.247	0.726	0.109	0.731	0.104	0.732	0.007	0.658	0.005
			0.8	1.000	0.980	0.867	0.997	0.198	0.999	0.200	0.997	0.040	0.997	0.017	0.998	0.935	0.532	0.913	0.204	0.912	0.197	0.902	0.008	0.839	0.004
			1	1.000	0.986	0.903	0.998	0.246	0.998	0.247	0.999	0.051	0.999	0.023	0.999	0.970	0.705	0.957	0.320	0.959	0.313	0.949	0.013	0.901	0.006
	0	0	0.2	0.132	0.109	0.084	0.106	0.050	0.103	0.045	0.089	0.044	0.061	0.047	0.075	0.060	0.038	0.054	0.030	0.055	0.026	0.043	0.040	0.025	0.034
			0.4	0.403	0.277	0.229	0.273	0.061	0.264	0.061	0.241	0.048	0.186	0.046	0.172	0.124	0.040	0.108	0.019	0.111	0.024	0.087	0.031	0.063	0.032
			0.6	0.865	0.651	0.592	0.648	0.119	0.646	0.128	0.608	0.068	0.520	0.065	0.372	0.252	0.080	0.230	0.022	0.245	0.030	0.206	0.028	0.156	0.038
			0.8	1.000	0.965	0.945	0.963	0.281	0.960	0.281	0.950	0.125	0.911	0.103	0.686	0.488	0.189	0.459	0.044	0.469	0.042	0.416	0.023	0.338	0.028
			1	1.000	1.000	1.000	0.999	0.469	1.000	0.470	0.999	0.197	0.997	0.162	0.916	0.750	0.397	0.694	0.094	0.701	0.099	0.669	0.038	0.566	0.038
		1	0.2	0.170	0.138	0.080	0.130	0.025	0.121	0.025	0.115	0.041	0.098	0.040	0.107	0.088	0.029	0.092	0.028	0.091	0.022	0.099	0.036	0.101	0.038
			0.4	0.547	0.389	0.269	0.373	0.031	0.359	0.030	0.326	0.035	0.315	0.035	0.264	0.194	0.039	0.214	0.026	0.197	0.025	0.214	0.030	0.231	0.034
			0.6	0.945	0.759	0.629	0.749	0.048	0.735	0.043	0.711	0.031	0.700	0.030	0.577	0.417	0.098	0.434	0.038	0.419	0.038	0.435	0.018	0.456	0.025
			0.8	1.000	0.980	0.941	0.972	0.074	0.972	0.079	0.964	0.031	0.960	0.023	0.879	0.706	0.261	0.717	0.059	0.704	0.057	0.718	0.012	0.733	0.016
			1	1.000	1.000	0.998	0.999	0.130	1.000	0.141	0.999	0.037	0.999	0.023	0.985	0.904	0.501	0.894	0.128	0.892	0.120	0.893	0.010	0.883	0.009
		$[1, t]'$	0.2	0.163	0.127	0.053	0.130	0.029	0.142	0.032	0.117	0.036	0.125	0.035	0.110	0.087	0.023	0.094	0.036	0.098	0.043	0.108	0.035	0.094	0.033
			0.4	0.529	0.344	0.192	0.337	0.037	0.357	0.034	0.323	0.033	0.346	0.031	0.291	0.203	0.037	0.214	0.046	0.220	0.050	0.235	0.027	0.205	0.026
			0.6	0.920	0.693	0.480	0.683	0.057	0.693	0.055	0.678	0.029	0.698	0.024	0.626	0.442	0.097	0.446	0.077	0.445	0.076	0.463	0.017	0.409	0.018
			0.8	1.000	0.944	0.823	0.938	0.096	0.943	0.098	0.935	0.031	0.942	0.020	0.919	0.743	0.227	0.727	0.111	0.743	0.121	0.739	0.010	0.646	0.010
			1	1.000	0.996	0.966	0.994	0.173	0.996	0.192	0.994	0.035	0.993	0.018	0.997	0.933	0.504	0.903	0.190	0.905	0.186	0.894	0.009	0.811	0.007
	0.5	0	0.2	0.106	0.100	0.087	0.091	0.050	0.094	0.049	0.077	0.049	0.049	0.044	0.062	0.053	0.035	0.049	0.033	0.055	0.033	0.036	0.039	0.024	0.038
			0.4	0.293	0.248	0.221	0.244	0.067	0.224	0.064	0.203	0.051	0.151	0.049	0.122	0.100	0.045	0.095	0.034	0.104	0.030	0.086	0.032	0.054	0.034
			0.6	0.702	0.585	0.544	0.563	0.125	0.559	0.126	0.530	0.069	0.443	0.060	0.283	0.218	0.076	0.203	0.033	0.217	0.033	0.185	0.029	0.141	0.037
			0.8	0.988	0.936	0.916	0.922	0.274	0.920	0.259	0.907	0.112	0.842	0.096	0.560	0.445	0.165	0.417	0.050	0.428	0.057	0.378	0.029	0.297	0.037
			1	1.000	1.000	0.999	0.999	0.456	0.999	0.471	0.998	0.183	0.991	0.138	0.846	0.704	0.343	0.654	0.094	0.667	0.103	0.622	0.039	0.519	0.037
		1	0.2	0.134	0.127	0.075	0.110	0.031	0.107	0.032	0.097	0.046	0.083	0.044	0.083	0.074	0.039	0.080	0.034	0.085	0.029	0.089	0.042	0.096	0.041
			0.4	0.392	0.340	0.241	0.314	0.037	0.299	0.038	0.280	0.043	0.255	0.044	0.198	0.168	0.058	0.197	0.032	0.187	0.034	0.197	0.040	0.204	0.037
			0.6	0.785	0.677	0.559	0.652	0.053	0.640	0.052	0.625	0.044	0.616	0.039	0.450	0.370	0.098	0.381	0.042	0.389	0.040	0.397	0.026	0.406	0.034
			0.8	0.989	0.947	0.893	0.939	0.083	0.935	0.086	0.930	0.045	0.912	0.033	0.762	0.641	0.217	0.654	0.070	0.657	0.072	0.675	0.019	0.665	0.024
			1	1.000	0.999	0.994	0.998	0.134	0.998	0.125	0.998	0.046	0.996	0.028	0.956	0.868	0.448	0.869	0.129	0.865	0.126	0.874	0.016	0.862	0.015
		$[1, t]'$	0.2	0.136	0.121	0.065	0.109	0.036	0.110	0.035	0.099	0.040	0.102	0.038	0.087	0.080	0.025	0.085	0.047	0.085	0.045	0.092	0.037	0.091	0.034
			0.4	0.371	0.306	0.180	0.277	0.042	0.290	0.045	0.273	0.042	0.280	0.039	0.203	0.175	0.042	0.181	0.058	0.198	0.056	0.206	0.030	0.179	0.033
			0.6	0.748	0.612	0.447	0.582	0.070	0.597	0.065	0.582	0.044	0.601	0.035	0.455	0.379	0.097	0.393	0.077	0.403	0.084	0.416	0.024	0.360	0.026
			0.8	0.979	0.895	0.780	0.874	0.116	0.885	0.106	0.878	0.042	0.890	0.029	0.783	0.656	0.204	0.671	0.126	0.685	0.122	0.690	0.019	0.602	0.018
			1	1.000	0.994	0.964	0.986	0.157	0.989	0.160	0.984	0.046	0.987	0.022	0.972	0.881	0.428	0.876	0.192	0.885	0.219	0.871	0.014	0.787	0.012
	0.9	0	0.2	0.104	0.101	0.080	0.088	0.052	0.090	0.053	0.075	0.051	0.054	0.050	0.060	0.052	0.033	0.052	0.032	0.054	0.034	0.040	0.040	0.020	0.037
			0.4	0.285	0.253	0.202	0.228	0.063	0.228	0.068	0.208	0.055	0.152	0.049	0.124	0.105	0.051	0.099	0.030	0.100	0.030	0.083	0.034	0.053	0.037
			0.6	0.684	0.596	0.524	0.549	0.123	0.543	0.125	0.514	0.071	0.431	0.065	0.277	0.221	0.079	0.212	0.041	0.208	0.040	0.177	0.030	0.130	0.033
			0.8	0.975	0.926	0.896	0.918	0.269	0.909	0.265	0.896	0.114	0.835	0.090	0.532	0.430	0.166	0.401	0.051	0.406	0.053	0.381	0.034	0.298	0.032
			1	1.000	0.999	0.998	0.999	0.428	0.999	0.439	0.998	0.176	0.989	0.136	0.817	0.686	0.351	0.664	0.098	0.664	0.103	0.620	0.035	0.528	0.038
		1	0.2	0.141	0.130	0.090	0.113	0.031	0.108	0.031	0.087	0.044	0.080	0.043	0.086	0.079	0.036	0.084	0.030	0.084	0.031	0.087	0.041	0.098	0.038
			0.4	0.380	0.330	0.251	0.297	0.035	0.284	0.038	0.274	0.047	0.259	0.043	0.192	0.169	0.052	0.182	0.032	0.180	0.037	0.189	0.035	0.203	0.036
			0.6	0.772	0.672	0.571	0.637	0.051	0.631	0.053	0.616	0.047	0.601	0.040	0.424	0.358	0.103	0.385	0.044	0.372	0.050	0.396	0.029	0.409	0.033
			0.8	0.985	0.942	0.897	0.934	0.082	0.925	0.085	0.925	0.049	0.914	0.034	0.740	0.631	0.227	0.659	0.071	0.645	0.072	0.665	0.020	0.678	0.023
			1	1.000	0.999	0.996	0.998	0.131	0.998	0.136	0.997	0.049	0.996	0.028	0.946	0.862	0.414	0.867	0.121	0.860	0.129	0.867	0.015	0.860	0.016
		$[1, t]'$	0.2	0.130	0.121	0.066	0.100	0.035	0.108	0.036	0.100	0.043	0.101	0.040	0.080	0.078	0.025	0.083	0.043	0.088	0.046	0.095	0.039	0.086	0.035
			0.4	0.345	0.294	0.183	0.258	0.043	0.278	0.044	0.255	0.047	0.272	0.040	0.197	0.175	0.046	0.188	0.059	0.187	0.060	0.203	0.035	0.180	0.030
			0.6	0.695	0.583	0.430	0.564	0.070	0.574	0.068	0.571	0.044	0.576	0.036	0.422	0.354	0.091	0.397	0.086	0.405	0.086	0.415	0.024	0.346	0.025
			0.8	0.962	0.886	0.770	0.860	0.111	0.876	0.106	0.862	0.047	0.878	0.029	0.756	0.645	0.202	0.670	0.128	0.681	0.118	0.687	0.018	0.576	0.018
			1	1.000	0.990	0.961	0.985	0.164	0.986	0.160	0.982	0.048	0.987	0.024	0.956	0.874	0.419	0.881	0.193	0.885	0.206	0.872	0.015	0.772	0.012

Note: Simulations are based on DGP (e). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are $Haar$ and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, $b - 1$ demonstrates the fractional integration order of $z_{2,t}$.

Table 13: Size Adjusted Power of the Cointegration tests under the Null Hypothesis $H_0 : r = 1$ vs Fractional Cointegration with Sample Size=1024

T	θ	δ_t	b	$d_1 = 0.1$											$d_1 = 1$										
				Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8	Λ	Λ^L Haar	Λ^{L*} Haar	Λ^L D_2	Λ^{L*} D_2	Λ^L S_2	Λ^{L*} S_2	Λ^L S_4	Λ^{L*} S_4	Λ^L S_8	Λ^{L*} S_8
1024	-0.9	0	0.2	0.749	0.625	0.616	0.927	0.781	0.917	0.755	0.922	0.305	0.913	0.081	0.611	0.415	0.106	0.402	0.097	0.402	0.089	0.417	0.170	0.391	0.063
			0.4	1.000	0.998	0.998	1.000	0.999	1.000	0.999	1.000	0.922	1.000	0.370	0.947	0.850	0.508	0.851	0.471	0.842	0.473	0.842	0.455	0.835	0.106
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.634	1.000	0.994	0.902	0.992	0.880	0.992	0.882	0.992	0.794	0.991	0.305
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.617	1.000	1.000	0.969	0.999	0.960	1.000	0.960	1.000	0.923	0.999	0.528
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.599	1.000	0.999	0.975	1.000	0.964	0.999	0.964	0.999	0.940	0.999	0.534
		1	0.2	0.890	0.780	0.837	0.995	0.748	0.996	0.756	0.995	0.135	0.994	0.029	0.818	0.625	0.168	0.621	0.226	0.618	0.222	0.614	0.023	0.624	0.016
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.431	1.000	0.082	0.999	0.974	0.725	0.974	0.756	0.975	0.761	0.973	0.061	0.975	0.010
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.506	1.000	0.112	1.000	1.000	0.985	1.000	0.986	1.000	0.987	1.000	0.174	1.000	0.012
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.528	1.000	0.113	1.000	1.000	0.995	1.000	0.994	1.000	0.994	1.000	0.287	1.000	0.032
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.513	1.000	0.109	1.000	1.000	0.993	1.000	0.995	1.000	0.995	1.000	0.288	1.000	0.033
		$[1, t]'$	0.2	0.937	0.840	0.931	1.000	0.925	1.000	0.935	1.000	0.183	1.000	0.039	0.955	0.810	0.275	0.813	0.462	0.806	0.471	0.803	0.050	0.818	0.018
			0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.522	1.000	0.107	1.000	0.999	0.883	0.999	0.945	0.999	0.948	0.999	0.122	0.999	0.016
			0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.568	1.000	0.119	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.300	1.000	0.025
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.563	1.000	0.126	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.367	1.000	0.038
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.598	1.000	0.126	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.376	1.000	0.045
	-0.5	0	0.2	0.302	0.212	0.145	0.226	0.084	0.215	0.097	0.210	0.070	0.207	0.046	0.119	0.101	0.031	0.094	0.017	0.101	0.018	0.102	0.076	0.095	0.054
			0.4	0.922	0.761	0.686	0.760	0.446	0.769	0.457	0.760	0.126	0.756	0.041	0.392	0.280	0.054	0.270	0.038	0.273	0.036	0.289	0.122	0.258	0.057
			0.6	1.000	0.999	0.999	1.000	0.988	0.999	0.987	1.000	0.465	0.999	0.111	0.855	0.682	0.264	0.683	0.225	0.672	0.234	0.688	0.278	0.661	0.078
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.447	0.997	0.976	0.782	0.970	0.745	0.972	0.739	0.969	0.686	0.964	0.169
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.619	1.000	0.999	0.963	0.999	0.953	1.000	0.952	1.000	0.918	0.999	0.387
		1	0.2	0.429	0.283	0.151	0.320	0.028	0.314	0.025	0.318	0.028	0.328	0.032	0.171	0.136	0.022	0.129	0.021	0.143	0.021	0.140	0.029	0.153	0.042
			0.4	0.992	0.918	0.833	0.930	0.227	0.928	0.237	0.920	0.034	0.924	0.016	0.593	0.438	0.065	0.428	0.090	0.430	0.082	0.448	0.019	0.447	0.026
			0.6	1.000	1.000	1.000	1.000	0.947	1.000	0.934	1.000	0.096	1.000	0.019	0.973	0.890	0.398	0.882	0.450	0.888	0.437	0.887	0.031	0.888	0.013
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.402	1.000	0.075	1.000	0.999	0.919	0.999	0.921	0.999	0.924	0.999	0.077	0.998	0.005
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.494	1.000	0.114	1.000	1.000	0.993	1.000	0.993	1.000	0.994	1.000	0.198	1.000	0.014
		$[1, t]'$	0.2	0.509	0.326	0.134	0.376	0.052	0.383	0.055	0.368	0.028	0.366	0.030	0.215	0.161	0.021	0.163	0.044	0.159	0.046	0.169	0.026	0.169	0.043
			0.4	0.997	0.943	0.824	0.953	0.344	0.956	0.354	0.953	0.037	0.953	0.016	0.702	0.511	0.069	0.515	0.178	0.512	0.174	0.530	0.025	0.536	0.026
			0.6	1.000	1.000	1.000	1.000	0.978	1.000	0.974	1.000	0.115	1.000	0.022	0.995	0.950	0.450	0.949	0.664	0.949	0.665	0.949	0.051	0.951	0.017
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.470	1.000	0.086	1.000	1.000	0.971	1.000	0.992	1.000	0.990	1.000	0.117	1.000	0.009
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.561	1.000	0.128	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.324	1.000	0.024
	0	0	0.2	0.175	0.159	0.130	0.163	0.077	0.154	0.076	0.151	0.080	0.141	0.050	0.089	0.085	0.039	0.088	0.031	0.087	0.037	0.091	0.061	0.082	0.052
			0.4	0.657	0.563	0.517	0.581	0.254	0.583	0.251	0.565	0.126	0.567	0.059	0.236	0.210	0.052	0.212	0.038	0.203	0.039	0.225	0.103	0.197	0.056
			0.6	0.998	0.989	0.983	0.987	0.850	0.988	0.852	0.987	0.345	0.987	0.101	0.603	0.523	0.165	0.537	0.125	0.524	0.135	0.545	0.191	0.524	0.081
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.864	1.000	0.268	0.955	0.904	0.565	0.912	0.531	0.899	0.523	0.910	0.455	0.895	0.136
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.580	1.000	0.998	0.925	0.998	0.903	0.998	0.916	0.998	0.864	0.997	0.258
		1	0.2	0.236	0.206	0.150	0.226	0.025	0.210	0.024	0.216	0.042	0.216	0.050	0.125	0.119	0.035	0.107	0.026	0.118	0.025	0.120	0.042	0.129	0.051
			0.4	0.843	0.740	0.668	0.758	0.092	0.748	0.093	0.751	0.042	0.760	0.039	0.363	0.322	0.059	0.332	0.061	0.325	0.062	0.334	0.029	0.351	0.042
			0.6	1.000	0.999	0.997	0.999	0.542	0.999	0.548	0.999	0.082	0.999	0.034	0.828	0.746	0.235	0.749	0.248	0.748	0.248	0.755	0.029	0.769	0.030
			0.8	1.000	1.000	1.000	1.000	0.998	1.000	0.998	1.000	0.239	1.000	0.051	0.997	0.985	0.727	0.986	0.730	0.986	0.748	0.985	0.066	0.987	0.016
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.436	1.000	0.087	1.000	1.000	0.983	1.000	0.982	1.000	0.982	1.000	0.143	1.000	0.009
		$[1, t]'$	0.2	0.265	0.229	0.133	0.229	0.037	0.229	0.034	0.228	0.040	0.230	0.052	0.122	0.114	0.028	0.128	0.039	0.128	0.038	0.134	0.042	0.135	0.052
			0.4	0.878	0.771	0.638	0.763	0.153	0.777	0.147	0.778	0.049	0.778	0.038	0.414	0.359	0.060	0.382	0.110	0.383	0.105	0.387	0.034	0.399	0.046
			0.6	1.000	1.000	0.997	0.999	0.682	0.999	0.671	1.000	0.095	0.999	0.033	0.892	0.810	0.253	0.827	0.403	0.823	0.411	0.823	0.054	0.834	0.032
			0.8	1.000	1.000	1.000	1.000	0.999	1.000	0.998	1.000	0.257	1.000	0.052	1.000	0.998	0.801	0.998	0.904	0.997	0.907	0.998	0.106	0.998	0.021
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.507	1.000	0.099	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.220	1.000	0.014
	0.5	0	0.2	0.166	0.159	0.133	0.152	0.072	0.148	0.079	0.142	0.074	0.137	0.059	0.087	0.086	0.046	0.088	0.041	0.084	0.038	0.087	0.063	0.078	0.053
			0.4	0.596	0.552	0.505	0.544	0.220	0.537	0.222	0.550	0.120	0.527	0.064	0.219	0.208	0.062	0.195	0.043	0.200	0.050	0.214	0.086	0.191	0.054
			0.6	0.988	0.974	0.966	0.979	0.780	0.981	0.790	0.979	0.334	0.977	0.107	0.536	0.493	0.175	0.508	0.125	0.504	0.119	0.508	0.156	0.499	0.080
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.828	1.000	0.280	0.916	0.877	0.522	0.890	0.470	0.883	0.476	0.891	0.352	0.880	0.129
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.532	1.000	0.996	0.902	0.997	0.881	0.997	0.892	0.998	0.790	0.996	0.271
		1	0.2	0.213	0.201	0.158	0.202	0.025	0.191	0.026	0.204	0.047	0.206	0.054	0.111	0.108	0.044	0.108	0.030	0.113	0.029	0.118	0.045	0.125	0.051
			0.4	0.751	0.697	0.636	0.724	0.084	0.696	0.076	0.719	0.051	0.724	0.048	0.323	0.305	0.070	0.317	0.061	0.323	0.060	0.328	0.032	0.331	0.047
			0.6	0.999	0.997	0.995	0.999	0.430	0.998	0.470	0.997	0.093	0.998	0.043	0.761	0.719	0.213	0.720	0.220	0.731	0.228	0.736	0.035	0.744	0.035
			0.8	1.000	1.000	1.000	1.000	0.985	1.000	0.987	1.000	0.236	1.000	0.065	0.989	0.978	0.686	0.979	0.666	0.980	0.637	0.979	0.065	0.982	0.024
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.489	1.000	0.109	1.000	1.000	0.972	1.000	0.973	1.000	0.970	1.000	0.135	1.000	0.013
		$[1, t]'$	0.2	0.232	0.219	0.120	0.206	0.038	0.212	0.035	0.207	0.048	0.210	0.055	0.117	0.114	0.037	0.124	0.041	0.116	0.041	0.126	0.045	0.128	0.052
			0.4	0.796	0.741	0.598	0.718	0.125	0.736	0.134	0.726	0.056	0.740	0.050	0.371	0.349	0.069	0.363	0.102	0.357	0.096	0.369	0.038	0.375	0.047
			0.6	1.000	0.999	0.993	0.998	0.562	0.998	0.542	0.998	0.110	0.999	0.043	0.823	0.776	0.258	0.794	0.341	0.799	0.350	0.801	0.051	0.809	0.039
			0.8	1.000	1.000	1.000	1.000	0.991	1.000	0.994	1.000	0.291	1.000	0.073	0.998	0.996	0.759	0.995	0.826	0.995	0.833	0.997	0.117	0.997	0.030
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.563	1.000	0.122	1.000	1.000	0.995	1.000	0.999	1.000	1.000	1.000	0.239	1.000	0.019
	0.9	0	0.2	0.142	0.137	0.118	0.149	0.070	0.153	0.080	0.140	0.071	0.140	0.057	0.086	0.084	0.038	0.085	0.038	0.082	0.037	0.092	0.064	0.080	0.055
			0.4	0.555	0.518	0.488	0.537	0.214	0.538	0.225	0.539	0.118	0.527	0.070	0.206	0.197	0.058	0.195	0.048	0.210	0.050	0.209	0.088	0.192	0.058
			0.6	0.987	0.977	0.972	0.981	0.771	0.976	0.760	0.978	0.320	0.977	0.106	0.532	0.499	0.158	0.506	0.129	0.498	0.125	0.524	0.170	0.489	0.086
			0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.814	1.000	0.271	0.911	0.880	0.514	0.883	0.447	0.880	0.452	0.888	0.344	0.877	0.118
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.550	0.999	0.996	0.893	0.996	0.876	0.996	0.878	0.999	0.768	0.995	0.245
		1	0.2	0.210	0.200	0.152	0.197	0.026	0.195	0.030	0.198	0.047	0.203	0.056	0.111	0.110	0.042	0.105	0.026	0.117	0.029	0.111	0.045	0.117	0.051
			0.4	0.741	0.702	0.631	0.715	0.083	0.695	0.085	0.710	0.051	0.717	0.049	0.326	0.312	0.069	0.312	0.059	0.321	0.057	0.320	0.035	0.333	0.049
			0.6	0.999	0.997	0.995	0.998	0.419	0.997	0.429	0.997	0.095	0.998	0.048	0.753	0.719	0.223	0.713	0.219	0.718	0.203	0.732	0.031	0.741	0.036
			0.8	1.000	1.000	1.000	1.000	0.980	1.000	0.980	1.000	0.247	1.000	0.068	0.986	0.976	0.666	0.975	0.654	0.979	0.646	0.982	0.066	0.981	0.022
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.484	1.000	0.116	1.000	1.000	0.971	1.000	0.962	1.000	0.960	1.000	0.119	1.000	0.012
		$[1, t]'$	0.2	0.221	0.210	0.119	0.210	0.040	0.209	0.041	0.208	0.047	0.207	0.059	0.118	0.116	0.035	0.120	0.045	0.122	0.042	0.128	0.043	0.137	0.053
			0.4	0.772	0.725	0.587	0.724	0.127	0.727	0.128	0.726	0.055	0.731	0.052	0.357	0.339	0.070	0.364	0.105	0.363	0.106	0.366	0.038	0.381	0.049
			0.6	0.999	0.997	0.991	0.998	0.526	0.998	0.545	0.998	0.111	0.998	0.050	0.816	0.778	0.244	0.790	0.321	0.789	0.341	0.801	0.052	0.809	0.041
			0.8	1.000	1.000	1.000	1.000	0.987	1.000	0.990	1.000	0.288	1.000	0.074	0.997	0.991	0.750	0.994	0.821	0.995	0.820	0.995	0.109	0.997	0.024
			1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.540	1.000	0.135	1.000	1.000	0.995	1.000	0.998	1.000	0.998	1.000	0.221	1.000	0.015

Note: Simulations are based on DGP (e). I skip the (d_1) notation in all test statistics, which share the same integration parameter d_1. Λ stands for Nielsen's (2010) test statistic for $H_0 : r = 1$ and $p = 3$. Similarly, Λ^L is my fractional wavelet variance ratio test (FWVR) for $H_0 : r = 1$ and $p = 3$, where Λ^{L*} is the wavestrapped version of the fractional wavelet variance ratio statistic (FWVRws) for the same null. Both FWVR and FWVRws utilize the same filters in their construction. These filters are *Haar* and D_2 (Dabuchies length 2 and 4 respectively), and S_2, S_4 and S_8 which are from the least asymetric wavelet family (Symlets) with lengths 4, 8 and 16 respectively. I use 10000 Monte Carlo simulations to obtain the critical values with T=1000. Furthermore, I consider three different adjustment procedures for the deterministic components. $\delta_t = 0$ is for the case of no deterministic adjustment, $\delta_t = 1$ denotes the case of only mean adjustment and $\delta_t = [1, t]'$ indicates the case of both mean and trend adjustment. Finally, $b - 1$ demonstrates the fractional integration order of $z_{2,t}$.