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# Fractional Seasonal Variance Ratio Unit Root Tests

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## Abstract

This paper introduces a non-parametric variance ratio testing procedure for seasonal unit roots by utilizing the fractional integration operator. This procedure includes unit root tests at zero, Nyquist, harmonic and joint frequencies. Different from the widely used seasonal unit root tests of Hylleberg et al. (1990)[HEGY], the proposed tests are free from any nuisance and tuning parameters. Furthermore, we develop a new bootstrap technique for the fractional seasonal variance ratio tests by utilizing wavelet filters. This technique allows the practitioners to test for the seasonal unit roots without estimating a parametric regression model. The Monte Carlo simulation evidence reveals that, our proposed fractional seasonal variance ratio [FSVR] tests and the wavelet based bootstrap counterparts have desirable size and power properties.

**Keywords:** Seasonal unit roots, Fractional integration, Wavelets, Wavestrapping
*JEL Clasification: C14, C22*

## 1. Introduction

Over the last three decades, in a growing body of empirical and theoretical studies scholars have investigated the presence of seasonal unit roots in the economic and financial data. In order to formally investigate this phenomenon, in their seminal paper, Hylleberg et al. (1990) [HEGY] extend the augmented Dickey-Fuller [ADF] unit root tests in the seasonal framework. The procedure in this paper is based on separate tests for unit roots at zero, Nyquist and harmonic frequencies. Other important developments in the seasonal unit root literature include Ghysels et al. (1994), Smith and Taylor (1998), Rodrigues and Taylor (2007), Smith et al. (2009) and del Barrio Castro et al. (2012) which propose extensions to the HEGY-type tests.

Despite their wide acceptance, the HEGY-type tests depend on the selection of the lag length parameter in the augmented regression to remove the impact of serial correlation in the innovations. In the literature, these type of parameters are referred to as tuning

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parameters. Furthermore, the selection of this tuning parameter affects the value of the test statistics but it is not reflected in the asymptotic distribution of the tests. However, Burridge and Taylor (2001) state that the dependence on the lag length parameter renders problems about the practical usage and the reliability of the HEGY-type tests. At this point, our point of departure from these testing methods is to develop a family of non-parametric seasonal unit root tests which are designed to get rid of any tuning and nuisance parameters from the test statistics and its asymptotic distribution.

In this regard, a serious attempt is made by Taylor (2005), who proposes a seasonal generalization to the non-parametric unit root test of Breitung (2002). Taylor (2005) designs the seasonal unit root test statistics such that the practitioners do not have to deal with a parametric specification for the innovation process. On the other hand, in the nonseasonal context, Nielsen (2009) presents a family of tuning parameter free unit root tests. In this paper, fractional integration techniques are used to construct a family of tests for an autoregressive unit root. Nielsen (2009) makes a decent contribution to the tests which are free of tuning parameters, such as Park and Choi (1988), Park (1990) and Breitung (2002), by improving asymptotic local power. To our knowledge, in the literature, the non-parametric fractional techniques do not exist for testing unit roots at the seasonal frequencies. Therefore, the first aim of our paper is to develop a family of non-parametric fractional seasonal unit root tests for different seasonal frequencies.

In the second part of our study, we aim to strengthen the proposed testing framework by developing a new wavelet based bootstrapping technique. In the recent literature, wavelets have gained an important place in the analysis of nonstationary data. In this context, Gençay et al. (2001) state that wavelet filtering provides a natural platform to deal with the non-stationarity found in the most economic and financial time series. Additionally, wavelets have many desirable features such as they can approximately decorrelate the time series processes (Percival and Walden, 2006). This feature makes wavelets useful tools also in bootstrapping. They can effectively create bootstrap replicates of the data without requiring the estimation of a parametric regression model to obtain i.i.d estimate of the innovations for re-sampling. This method is known as *wavestrapping*.

Recently, Trokić (2016) have utilized wavestrapping for unit root tests, Eroğlu (2016) have applied this method in cointegration framework and Li and Shukur (2013) have proposed a variant of wavelet based bootstrapping in panel unit root tests. In accordance with these studies, we propose a new wavestraping procedure for FSVR tests. In this procedure, we adopt a parallel application of the wavestrapping routine to the innovations of the observed data in each season. After successfully wavestrapping the data, we use the bootstrap replications in the FSVR tests.

The rest of the paper is organized as follows. In Section 2, the seasonal unit root model, the seasonal unit root hypothesis and the augmented HEGY-type tests are presented. Our nonparametric FSVR testing framework is developed in Section 3, and the asymptotic results are given in Section 4, respectively. In Section 5, a wavelet based bootstrap technique for the proposed testing procedure is introduced. The finite sample simulation study is presented in Section 6. Section 7 concludes the paper. The proofs of the theorems and lemmas, the technical details about wavelet filters and the tables containing the critical values and simulation results are placed in the appendix.

In what follows, we use the following notation: $\xrightarrow{D}$ to denote weak convergence, as sample size diverges to positive infinity; $\lfloor . \rfloor$ to denote the integer part of the argument; $a := b$ to indicate that $a$ is defined by $b$; $Re(.)$ and $Im(.)$ to denote real and imaginary parts of their arguments, respectively. Finally we define $\mathrm{i} = \sqrt{-1}$.

## 2. The Seasonal Unit Root Framework

### *2.1. The Seasonal Unit Root Model*

We consider the univariate model defining the seasonal time series $\{x_{St+s}\}$ with the following data generating process (DGP) [4]

$$\alpha(L)x_{St+s} = u_{St+s}, \qquad s = 1-S, ..., 0, \qquad t = 1, 2, ..., N \tag{1}$$

$$u_{St+s} = \phi(L)\epsilon_{St+s} \tag{2}$$

where $S$ is the number of seasons, $\alpha(L) = 1 - \sum_{i=1}^{S} \alpha_j L_j$ is an $S$ order AR polynomial which determines the seasonal unit root. Further, we assume stationary innovations with the serial correlation structure governed by AR polynomial $\phi(L) = 1 - \sum_{i=1}^{\infty} \phi_j L_j$. The total number of observations is given by $T = SN$. $N$ denotes the total number of observation in each season. For simplicity, we suppose same number of observations in these seasons. We assume that the initial conditions $x_{1-S}, ..., x_0$ are $o_p(T^{1/2})$. Following del Barrio Castro et al. (2012), we propose the following assumptions on the innovation sequence $\{\epsilon_{St+s}\}$ which further characterize the dynamics of seasonal model:

**Assumption.** $\mathcal{A}.1$ *The lag polynomial* $\phi(L) \neq 0$ *for all* $|L| \leq 1$, *and* $\sum_{j=0}^{\infty} j^h |\phi_j| < \infty$ *for some* $h \geq 1$.
$\mathcal{A}.2$ $\mathbb{E}[\epsilon_{St+s}] = \sigma^2$
$\mathcal{A}.3$ $\frac{1}{N}\sum_{t=1}^{N} \epsilon_{St+s} \xrightarrow{p} \sigma^2$ *for all* $s = 0, 1, ...., 1-S$
$\mathcal{A}.4$ $\mathbb{E}|\epsilon_{St+s}|^r < K < \infty$ *for some* $r \geq 4$.

These assumptions are very standard in the seasonal unit root literature. Assumption $\mathcal{A}.1$ implies stationarity of the innovation term $\epsilon_{St+s}$. In other words, the only source of nonstationary dynamics is the seasonal AR polynomial $\alpha(L)$. $\mathcal{A}.2$ and $\mathcal{A}.3$ indicates the error terms are homoscedastic. Note that this condition is strong enough to exclude

[4]The notation closely follows del Barrio Castro et al. (2012)

periodic or seasonal variance difference for the error terms. $\mathcal{A}.4$ allows the $r$th moment of $u_{St+s}$ to exist and to be bounded uniformly in $t$. The detailed discussion about the given assumption can be found in del Barrio Castro et al. (2012).

### 2.2. The Seasonal Unit Root Hypothesis

This section will follow del Barrio Castro et al. (2012) closely. Under the null hypothesis, we assume a seasonal unit root at each season S as follows:

$$H_0 : \alpha(L) = (1 - L^S) =: \Delta_S \tag{3}$$

This hypothesis can be partitioned as $H_0 = \cap_{j=0}^{\lfloor S/2 \rfloor} H_{0,j}$, where

$$\begin{aligned} &H_{0,i} : \alpha_i = 1, \quad i = 0, S/2 \\ &H_{0,j} : \alpha_j = 1 \ and \ \beta_j = 0 \quad j = 1, ..., S^* \end{aligned}$$

where $S^* = \lfloor (S-1)/2 \rfloor$. The seasonal lag polynomial can be decomposed as $\alpha(L) = \prod_{j=0}^{\lfloor S/2 \rfloor} \omega_j(L)$ with $\omega_0 = (1 - \alpha_0 L)$ (unit root in zero frequency), $\omega_j(L) = [1 - 2(\alpha_j cos\omega_j - \beta_j sin\omega_j)L + (\alpha_j^2 + \beta_j^2)L^2]$ ( $\omega_j = 2\pi j/S$ are seasonal frequencies) for $j = 1, ..., S^*$ and $\omega_{\lfloor S/2 \rfloor}(L) := \big(1 + \alpha_{\lfloor S/2 \rfloor} L\big)$.

The alternative hypothesis is $H_1 = \cup_{j=0}^{\lfloor S/2 \rfloor} H_{1,j}$ which ensures stationarity at one or more of the zero frequency or seasonal frequencies. We can explicitly write the each alternatives as:

$$\begin{aligned} &H_{1,i} : |\alpha_i| < 1, \quad i = 0, S/2 \\ &H_{0,j} : \alpha_j^2 + \beta_j^2 < 1 \ and \ \beta_j = 0 \quad j = 1, ..., S^* \end{aligned}$$

We can test each hypothesis separately to reveal the individual unit root characteristics at different frequencies. In addition, this setup also allows us to conduct joint testing for the above hypothesis.

### 2.3. The Augmented HEGY tests

We first discuss the parametric HEGY tests. Hylleberg et al. (1990) expand the AR(p+S) polynomial $\phi^*(z) = \alpha(z)\phi(z)$ around the zero and seasonal frequency unit roots and obtain the auxiliary HEGY regression,

$$\begin{aligned} \Delta_S x_{St+s} = \pi_0 x_{0,St+s-1} + \pi_{S/2} x_{S/2,St+s-1} + \sum_{i=1}^{S*} (\pi_i x_{i,St+s-1} + \pi_i^* x_{i,St+s-1}^*) \\ + \sum_{j=1}^{p} \phi_j^* \Delta_S x_{Sn+s-j} + \varepsilon_{Sn+s} \end{aligned} \tag{4}$$

where the independent variables are defined as

$$x_{0,St+s} := \sum_{j=0}^{S-1} x_{St+s-j}$$

$$x_{S/2,St+s} := \sum_{j=0}^{S-1} cos[(j+1)\pi]x_{St+s-j}$$

$$x_{i,St+s} := \sum_{j=0}^{S-1} cos[(j+1)\omega_i]x_{St+s-j} \qquad \textit{for all } i = 1, ..., S^*$$

$$x^*_{i,St+s} := \sum_{j=0}^{S-1} sin[(j+1)\omega_i]x_{St+s-j} \qquad \textit{for all } i = 1, ..., S^*$$

After estimating the model in equation (4), HEGY device the tests for the null hypotheses of unit root at zero, Nyquist and harmonic seasonal frequencies which are stated respectively as:

$$H_{0,k} : \pi_k = 0, \;\; k = 0, S/2 \tag{5}$$

$$H_{0,i} : \pi_i = \pi^*_i = 0, i = 1, ..., S^* \tag{6}$$

The tests for the presence of a unit root defined in Hylleberg et al. (1990) consist of the regression t statistic $t_k$ (left sided) for the exclusion of $x_{k,St+s-1}$ for $k = 0, S/2$ and $F_i$ for the joint exclusion of $x_{i,St+s-1}$ and $x^*_{i,St+s-1}$ for all $i = 1, ..., S^*$ from (4). Moreover, Ghysels et al. (1994) improve the HEGY-type tests by proposing the joint frequency (upper-tail) F-tests by using the same regression model in (4). The F test $F_{1,...,\lfloor S/2 \rfloor}$ is used for testing the exclusion of $\{x_{i,St+s-1}\}_{i=1}^{S^*}$ , $\{x^*_{i,St+s-1}\}_{i=1}^{S^*}$ and $x_{S/2,St+s-1}$, and $F_{0,...,\lfloor S/2 \rfloor}$ is designed for testing the exclusion of $x_{0,St+s-1}$, $\{x_{i,St+s-1}\}_{i=1}^{S^*}$ , $\{x^*_{i,St+s-1}\}_{i=1}^{S^*}$ and $x_{S/2,St+s-1}$. The null hypothesis for these two F tests can be represented respectively, as following:

$$H_{1,...,\lfloor S/2 \rfloor} = \bigcap_{i=1}^{\lfloor S/2 \lfloor} H_{0,i} : \;\; \pi_1 = \cdots = \pi_{S^*} = \pi_{\lfloor S/2 \rfloor} = 0$$

$$H_{0,...,\lfloor S/2 \rfloor} = \bigcap_{i=0}^{\lfloor S/2 \lfloor} H_{0,i} : \;\; \pi_0 = \pi_1 = \cdots = \pi_{S^*} = \pi_{\lfloor S/2 \rfloor} = 0$$

**Remark 1.** *Empirical implementation of HEGY-type tests forces the practitioners to choose lag order p. This procedure has a time complexity. Moreover, the finite sample performance depends on the choice of p, though the lag order is not reflected in the limiting distribution of the statistics. Although optimal lag length selection methods are discussed in the literature, such procedures may still work poorly in small samples. For a detailed discussion of lag length selection techniques, we refer the reader to del Barrio Castro et al. (2016). In our test, we will avoid these issues and propose fully nonparametric family of tests.*

## 3. Fractional Seasonal Variance Ratio Tests

In this section, we propose a new method for testing seasonal unit roots which is a family of nonparametric and tuning parameter-free tests indexed by the fractional parameter $d$. This method is a seasonal generalization of the powerful test of Nielsen (2009).

We first define the S dimensional process $X_t$ which separates the observed series $x_{St+s}$ from (1) into sub-seasonal observations. That is, each vector in this process corresponds to the observation in each season:

$$X_t := [x_{St-(S-1)}, x_{St-(S-2)}, ..., x_{St}]' \quad t = 0, ..., N$$

In our analysis, we will utilize the fractional integration operator of type II. This operator takes the weighted sum of the past values of a time series process, say $y_t$, where weights are determined by the fractional binomial coefficients. The following equation summarizes this operator:

$$\tilde{y}_t := \Delta_+^{-d_1} y_t = (1-L)_+^{-d_1} y_t = \sum_{k=0}^{t-1} \frac{\Gamma(k+d_1)}{\Gamma(d_1)\Gamma(k+1)} y_{t-k} = \sum_{k=0}^{t-k} \pi_k(d_1) x_{t-k} \tag{7}$$

where $1 \geq d_1 > 0$. We will apply this operator to each element of the $X_t$ separately to obtain:

$$\tilde{X}_t = [\Delta_+^{-d_1} x_{St-(S-1)}, \Delta_+^{-d_1} x_{St-(S-2)}, ..., \Delta_+^{-d_1} x_{St}]' \tag{8}$$

After obtaining $\tilde{X}_t$, we reconstruct the associated fractional transformation of the seasonal process from this process. In other words, we create a new variable, say $\tilde{x}_{St+s}$ from $\tilde{X}_t$ as each vector of $\tilde{X}_t$ corresponds to one season of the final process $\tilde{x}_{St+s}$.

**Remark 2.** *Applying the fractional integration operator as discussed above is new to the literature. For instance, Nielsen (2009) applies this operator to observed series directly. We do not follow Nielsen (2009) in this study purely because of technical reasons.*

Now consider the HEGY transformation of the fractionally integrated process $\tilde{x}_{St+s}$:

$$\begin{aligned}
\tilde{x}_{0,St+s} &:= \sum_{j=0}^{S-1} \tilde{x}_{St+s-j} \\
\tilde{x}_{S/2,St+s} &:= \sum_{j=0}^{S-1} cos[(j+1)\pi]\tilde{x}_{St+s-j}
\end{aligned} \tag{9}$$

$$\tilde{x}_{i,St+s} := \sum_{j=0}^{S-1} cos[(j+1)\omega_i]\tilde{x}_{St+s-j} \qquad for\ all\ i = 1, ..., S^*$$

$$\tilde{x}^*_{i,St+s} := \sum_{j=0}^{S-1} sin[(j+1)\omega_i]\tilde{x}_{St+s-j} \qquad for\ all\ i = 1, ..., S^*$$

With these objects we can define the test statistics for testing unit root hypothesis in different frequencies. First, consider the testing unit root on zero and Nyquist frequencies:

$$\tau_k(d_1) = N^{2d_1} \frac{\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{k,St+s}}{\sum_{t=1}^{N}\sum_{s=1-S}^{0} \tilde{x}^2_{k,St+s}} \ for\ k = 0, S/2 \tag{10}$$

**Remark 3.** *This test corresponds to the test of HEGY with the null hypothesis $H_{0,k}$ for $k = 0, S/2$.*

For the harmonic frequencies, we propose two variance ratio test statistic.

$$\tau_j(d_1) = N^{2d_1} \frac{\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s}}{\sum_{t=1}^{N}\sum_{s=1-S}^{0} \tilde{x}^2_{j,St+s}} \ for\ j = 1, .., S^* \tag{11}$$

$$\tau^*_j(d_1) = N^{2d_1} \frac{\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^{*2}_{j,St+s}}{\sum_{t=1}^{N}\sum_{s=1-S}^{0} \tilde{x}^{*2}_{j,St+s}} \ for\ j = 1, .., S^* \tag{12}$$

where $\tau_j(d_1)$ is the test statistic for the unit root hypothesis at the harmonic frequency $\omega_j$ for all $j = 1, ..., S^*$, $\tau^*_j(d_1)$ is the test statistic for the unit root hypothesis at the harmonic frequency $2\pi - \omega_j$ for all $j = 1, ..., S^*$.

**Remark 4.** *These tests correspond to the tests of HEGY with the null hypothesis $H_{0,j}$ for $j = 1, ..., S^*$.*

Notice that the unit roots in the harmonic frequencies are complex conjugates. Then, one can use the following statistic to jointly test the presence of the pairs of complex unit roots:

$$\tilde{\tau}_j(d_1) = (\tau_j(d_1) + \tau^*_j(d_1))/2 \ for\ j = 1, .., S^* \tag{13}$$

Finally, we derive the F type statistics for the joint test for the unit roots in the different frequencies:

$$\tau_{1,..,S/2}(d_1) = \frac{1}{S-1}\left(\tau_2(d_1) + \sum_{j=1}^{S^*}\tau_j(d_1) + \sum_{j=1}^{S^*}\tau^*_j(d_1)\right) \tag{14}$$

$$\tau_{0,..,S/2}(d_1) = \frac{1}{S}\left(\tau_0(d_1) + \tau_2(d_1) + \sum_{j=1}^{S^*}\tau_j(d_1) + \sum_{j=1}^{S^*}\tau^*_j(d_1)\right) \tag{15}$$

**Remark 5.** *Remark that the test in equation (14) corresponds to the joint unit root tests at all frequencies except the zero frequency. Further, the test in equation (15) corresponds to the joint unit root tests at all frequencies.*

## 4. Asymptotic Results for the Fractional Seasonal Variance Ratio Test

In this section, we present the asymptotic results for the FSVR tests. To derive the asymptotic results, we adopt a similar structure as del Barrio Castro et al. (2012). Under the overall null hypothesis, we can write $\{x_{St+s}\}$ as in $S$ dimensional vector form:

$$X_t = X_{t-1} + U_t \tag{16}$$

$$X_t := [x_{St-(S-1)}, x_{St-(S-2)}, ..., x_{St}]' \quad t = 0, ..., N \tag{17}$$

$$U_t := [u_{St-(S-1)}, u_{St-(S-2)}, ..., u_{St}]' \quad t = 0, ..., N \tag{18}$$

where the innovation process can be written as:

$$U_t = \sum_{j=0}^{\infty} \Phi_j E_{t-j}$$

$$E_t := [\epsilon_{St-(S-1)}, \epsilon_{St-(S-2)}, ..., \epsilon_{St}]' \quad t = 0, ..., N$$

Here $\Phi_0$ and $\Phi_j$s are defined as in del Barrio Castro et al. (2012):

$$\Phi_0 := \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ \phi_1 & 1 & 0 & ... & 0 \\ \phi_2 & \phi_1 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{S-1} & \phi_{S-2} & \phi_{S-3} & ... & 1 \end{bmatrix}$$

$$\Phi_j := \begin{bmatrix} \phi_{jS} & \phi_{jS-1} & \phi_{jS-2} & ... & \phi_{jS-(S-1)} \\ \phi_{jS+1} & \phi_{jS} & \phi_{jS-2} & ... & \phi_{jS-(S-2)} \\ \phi_{jS+2} & \phi_{jS} & \phi_{jS-1} & ... & \phi_{jS-(S-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{jS+S-1} & \phi_{jS+S-2} & \phi_{jS+S-3} & ... & \phi_{jS} \end{bmatrix} \quad j = 1, 2, ...$$

The following lemma can be found as Lemma 1 in del Barrio Castro et al. (2012):

**Lemma 1.** *Let $X_t$ is generated as in equations (16)-(18). Then under Assumptions $\mathcal{A}.1-4$,*

$$N^{-1/2} X_{\lfloor rN \rfloor} \xrightarrow{D} \sigma \Phi(1) W(t) \quad r \in [0, 1]$$

*where $W(r)$ is $S$ dimensional standard Brownian motion process, $\Phi(1) := \sum_{j=0}^{\infty} \Phi_j$, the right element of above convergence can be expressed as*

$$(1/S)\sigma \left[ \phi(1)C_0 + \phi(-1)C_{S/2} + 2\sum_{i=1}^{S^*}(b_i C_i + a_i C_i^*) \right] W(r)$$

where $a_i = Im(\phi[\exp(\mathrm{i}w_i)])$, $b_i = \mathcal{R}e(\phi[\exp(\mathrm{i}w_i)])$ for $i = 1, ..., S^*$, $C_j$ for all $j = 0, ..., S/2$ and $C_k^*$ for $k = 1, ...S^*$ are circulant matrices. Further information about these matrices can be found in Appendix A.

The following lemma demonstrates the limiting distribution of the fractionally transformed process:

**Lemma 2.** *Let $X_t$ is generated as in equations (16)-(18) and $\tilde{X}_t$ is obtained from equation (8). Under Assumptions $\mathcal{A}.1-4$, then*

$$N^{d_1-1/2}\tilde{X}_{\lfloor rN\rfloor} \xrightarrow{D} \sigma\Phi(1)W_{d_1+1}(t) \quad r\in[0,1]$$

*where*

$$W_{d_1+1}(r) = \frac{1}{\Gamma(d_1+1)}\int_0^r (r-t)^{d_1}dW(t) \tag{19}$$

*is $S$ dimensional Fractional Brownian motion.*

Using the objects derived in Lemma 1 and Lemma 2, we reach the following asymptotic results for the FSVR test statistics:

**Theorem 1.** *Let the assumptions of Lemma 1 hold. Consider the FSVR test statistics given by equations (10)-(15). As $N\to\infty$,*

$$\begin{aligned}
\tau_i(d_1) &\xrightarrow{D} \frac{\int_0^1 W_i(r)^2dr}{\int_0^1 W_{d_1+1,i}(r)^2dr} = \eta_i(d_1) \quad i=0,S/2\\
\tau_j(d_1) &\xrightarrow{D} \frac{\int_0^1 W_j(r)^2dr + \int_0^1 W_j^*(r)^2dr}{\int_0^1 W_{d_1+1,j}(r)^2dr + \int_0^1 W_{d_1+1,j}^*(r)^2dr} = \eta_j(d_1) \textit{ for } j=1,...,S^*\\
\tau_j^*(d_1) &\xrightarrow{D} \eta_j(d_1) \textit{ for } j=1,...,S^*\\
F_j(d_1) &\xrightarrow{D} \eta_j(d_1) \textit{ for } j=1,...,S^*\\
\tau_{1,..,S/2}(d_1) &\xrightarrow{D} \frac{1}{S-1}\left(\eta_2(d_1)+2\sum_{j=1}^{S^*}\eta_j(d_1)\right)\\
\tau_{0,..,S/2}(d_1) &\xrightarrow{D} \frac{1}{S}\left(\eta_0(d_1)+\eta_2(d_1)+2\sum_{j=1}^{S^*}\eta_j(d_1)\right)
\end{aligned}$$

*where $W_0(r)$, $W_{S/2}(r)$, $W_j^*(r)$ and $W_j(r)$ for $j=1,...,S^*$ are the independent standard Brownian motions, $W_{d_1+1,0}(r)$, $W_{d_1+1,S/2}(r)$, $W_{d_1+1,j}^*(r)$ and $W_{d_1+1,j}(r)$ are the independent fractional Brownian motions for $j=1,...,S^*$, respectively.*

**Remark 6.** *The convergence results stated in Theorem 1 do not take the deterministic component into account. To adjust for the deterministic elements, we extend these results by the following generalization of the DGP in (1)-(2):*

$$y_{St+s} = \mu_{St+s} + x_{St+s}, s=1-S,...,0, t=1,...,N \tag{20}$$

*where $x_{St+s}$ is defined in (1)-(2) and $\mu_{St+s} = \gamma' Z_{St+s}$ where $Z_{St+s}$ are purely deterministic. Following Smith et al. (2009), we define the characterization of $\mu_{St+s}$ with a typology of three cases, namely: no deterministic component ($j = 0$), seasonal intercepts ($j = 1$), seasonal intercepts and seasonal trends ($j = 2$). When the deterministic component is involved, to obtain tests which are exact invariant to the elements of $\gamma$, our test statistics (10)-(15) should be redefined using de-trended series as $\hat{x}_{St+s} = x_{St+s} - \hat{\gamma}' Z_{St+s}$. The detrending in this paper is done by using OLS detrending as in, Smith et al. (2009) and Nielsen (2009). For example, we have the demeaned or detrended analogues of the Brownian motion in the asymptotic distribution of unit root test at the zero frequency as follows:*

$$B_{0,d_1+1,j}(r) = W_{0,d_1+1}(r) - \left(\int_0^1 W_0(t)D_j(t)dt\right)\left(\int_0^1 D_j(t)D_j(t)dt\right)^{-1} \times \int_0^r \frac{1}{\Gamma(d_1)}(r-t)^{d_1-1}D_j(t)dt \quad \forall\ j = 1,2$$

*where $D_1(t) = 1$ and $D_2(t) = [1,t]'$. For other frequencies this object can be obtained similarly. For the demeaning and the detrending procedures for the HEGY type tests, see Smith et al. (2009).*

**Remark 7.** *The asymptotic results in Theorem 1 are based on the seasonal unit root null hypothesis of $H_0$ given at equation (3). Under the local alternatives, Rodrigues and Taylor (2007) present the decomposition of $\alpha(L)$ presented in Section 2.2 as $\alpha_i = (1 + \nu_i/T)$ for $i = 0, S/2$, and $\alpha_j = (1 + \nu_j/T), \beta_j = 0$ for $j = 1, ..., S^*$, where $\nu_0, \nu_1, ...., \nu_{S/2}$ are finite constants. Therefore, following Rodrigues and Taylor (2007) and Nielsen (2009), under the local alternatives, the Brownian motions in Theorem 1 can be replaced by the corresponding Ornstein-Uhlenbeck processes which are given as follows:*

$$J_{j,c}(r) = \int_0^r exp(-\nu_j(r-\lambda))dW_j(t)$$

$$\tilde{J}_{j,c,d_1+1}(r) = \int_0^r exp(-\nu_j(r-\lambda))dW_{j,d_1+1}(t)$$

*where $\nu_j$, $j = 0, ..., S/2$ are the frequency specific non-centrality parameters.*

**Remark 8.** *Let $CV_{\gamma,S}(\eta_j(d_1))$ be the $\gamma$-th quantile of the asymptotic distribution of the test statistic $\tau_j(d_1)$ for $j = 0, ..., S/2$. We reject the one sided t type test if $\tau_j(d_1) < CV_{\gamma,S}(\eta_j(d_1))$ for all $j = 0, ..., S/2$ . For the joint tests, we reject the null hypothesis if $\tau_{0,...,S/2}(d_1) > CV_{(1-\gamma),S}(\eta_{0,...,S/2}(d_1))$ where $CV_{(1-\gamma),S}(\eta_{0,...,S/2}(d_1))$ is the $(1-\gamma)$-th quantile of the empirical bootstrap distribution of the test statistic $\tau_{0,...,S/2}(d_1)$.*

*The asymptotic critical values for the FSVR tests for three different parameter values for $d_1$ and two different number of seasons $S$ are given in Table 1. We report the critical values for $d_1 \in \{0.1, 0.5, 1\}$. Moreover, we consider the cases $S \in \{4, 12\}$ which are empirically the most relevant ones. $S = 4$ corresponds to quarterly data and $S = 12$ is* *used for monthly data. In this table, the critical values are also classified according to the deterministic structures given in Remark 6 and the nominal significance levels.*

## 5. Wavelet Based Bootstrapping

In this section, we introduce a nonparametric bootstrapping method for the FSVR tests. This method is known as 'wavestrapping' and utilizes wavelet filters[5]. The use of wavelet filters removes the necessity of the estimation of a parametric regression model for resampling from innovations. This is due to the fact that Wavelet filters can decompose the observed series into asymptotically decorrelated components (Percival and Walden, 2006). After obtaining the decorrelated components from the wavelet decomposition, we can simply resample them to generate bootstrap replicates. Since wavelet filters have perfect reconstruction property, we can get the bootstrapped data from resampled coefficients.

In the recent lieterature, the wavelet based bootstrapping methods are used in nonstationary data analysis. For instance, Trokić (2016) utilizes wavestrapping for unit root tests, Eroglu (2016) applies this method in cointegration framework and Li and Shukur (2013) propose a variant of wavestrapping in panel unit root tests.

Different from the authors mentioned above, we develop a new technique using wavestraping in this paper. This new technique composes of a parallel application of the wavestraping to the innovations in each seasons. The following routine summarizes the wavestrapping for seasonal unit root tests.

1. From the observed series $x_{St+s}$, obtain the FSVR test statistics $\tau_i(d_1)$ for $i = 0, ..., S/2$, $\tilde{\tau}_i(d_1)$ for $i = 1, ..., S^*$, $\tau_{0,..,S/2}(d_1)$ and $\tau_{1,..,S/2}(d_1)$.
2. Pick $B$ as the number of bootstrap replications and also choose a suitable wavelet[6].
3. We apply wavestrapping under the null hypothesis as other bootstrapping rotines. Obtain the seasonal difference of the observed series (or the innovations), $\Delta_S x_{St+s} = u_{St+s}$.
4. From these innovations $u_{St+s}$, obtain the seasonal vector form,

$$U_t = [u_{St-(S-1)}, u_{St-(S-2)}, ..., u_{St}]'$$

5. Apply $l$ level[7] multisignal wavelet transform to the seasonal innovations vector $U_t$. This decomposition yields a $N/2^l \times S$ matrix of the approximation coefficients $V_l$

[5] Wavelet filters are time-frequency domain tools which can separate the data into different frequency components, which are called as wavelet coefficients. A broad discussion about wavelet filters is presented in Gençay et al. (2001)

[6] For the wavelet selection, we suggest to choose 'haar' wavelet, since it provides best results in our simulations and in other wavestrapping exercise in the literature.

[7] We suggest use $l = J - 2$ where $J$ is the closest dyadic power of seasonal simple size N

and $l$ matrices of the wavelet coefficients, $W_1, W_2, ...$ and $W_l$. Note that each wavelet coefficient matrix $W_j$ has dimension of $N/2^j \times S$.

6. Now, for $b = 1, ..., B$ define $i_j^{*(b)}$ as the resampled index for the wavelet coefficient matrix $W_j$ for all $j = 1, ..., l$. We get the resampled wavelet coefficient matrices as $W_j^{*(b)} = W_j(i_j^{*(b)}, 1 : S)$, where the notation $Z(a, b)$ indicates the row index of the matrix $Z$ is denoted by $a$ and the column index of the matrix Z is denoted as $b$. Note that we do not apply the resampling scheme to the approximation coefficients $V_{l,t}$[8].
7. By using the reconstruction filter, we obtain the wavestrapped matrices of the seasonal observations $U_t^{*(b)}$. After the integration of $U_t^{*(b)}$, we obtain $X_t^{*(b)} = X_{t-1}^{*(b)} + U_t^{*(b)}$. Finally unstacking $X_t^{*(b)}$, we get $x_{St+s}^{*(b)}$ which is wavestrapped version of the observed series.
8. Applying the same testing procedure to the wavestrapped series $x_{St+s}^{*(b)}$, we obtain the wavestrapped tests $\tau_j^{*(b)}(d_1)$ for all $j = 0, ..., S/2$, $\tilde{\tau}_i^{*(b)}(d_1)$ for all $i = 1, ..., S^*$ and joint tests $F_{1,...,S/2}^{*(b)}(d_1)$ and $F_{0,...,S/2}^{*(b)}(d_1)$.
9. Repeating the steps 6-8, we get the empirical bootstrap distribution for these tests. Using this empirical distribution, we can apply standard bootstrap inference.

**Remark 9.** *Let $CV_{\gamma,S}(\tau_j^{**}(d_1))$ be the $\gamma$-th quantile of the empirical bootstrap distribution of the test statistic $\tau_j(d_1)$ for $j = 0, ..., S/2$. We reject the one sided t type test if $\tau_j(d_1) < CV_{\gamma,S}(\tau_j^{**}(d_1))$ for all $j = 0, ..., S/2$ . For the joint tests, we reject the null hypothesis if $\tau_{0,...,S/2}(d_1) > CV_{(1-\gamma),S}(\tau_{0,...,S/2}^{**}(d_1))$ where $CV_{(1-\gamma),S}(\tau_{0,...,S/2}^{**}(d_1))$ is the $(1-\gamma)$-th quantile of the empirical bootstrap distribution of the test statistic $\tau_{0,...,S/2}(d_1)$.*

**Remark 10.** *In each Monte Carlo iteration, we apply the above bootstrap algorithm. This type of setups is known as bootstrap in bootstrap or Double bootstrap (see Davidson and MacKinnon (2007)). Unfortunately, this procedure is computationally intensive. In order to reduce the computation time, we apply the Fast Double bootstrap algorithm of Davidson and MacKinnon (2007) which is appropriate if the test statistic is pivotal as in our case. In this algorithm, we simply set $B = 1$ and obtain 1 bootstrapped test statistic for each Monte Carlo replication. As a result, we have MC number of bootstrapped test statistic, which can be replaced by the empirical bootstrap distribution of the associated test statistic.*

**Remark 11.** *Cavaliere, Cavaliere et al. (2017) propose a similar bootstrapping routine by sampling the innovations for each season at the same time with same index. However, they differ from us since they use parametric HEGY regression to obtain innovations and also use wild bootstrapping. In our study, thanks to wavelet filters, we can directly use seasonally differenced data for resampling without estimating any parameters.*

[8]This is the common practice in wavestrapping routine. Resampling the approximation coefficients is possible but it may provide poorer inference according to the unreported simulations

## 6. Finite Sample Simulations

In this section, we investigate the finite sample size and size-adjusted power of the conventional HEGY tests from section 2.3, the FSVR tests from section 3, and the corresponding wavelet based bootstrap FSVR tests from section 5. The experiments are done using 10,000 Monte Carlo replications. The reported finite sample simulations in this section are based on the following quarterly ($S = 4$) DGP:

$$x_{4t+s} = (1 - c/N)x_{4t+s-1} + u_{4t+s}, \qquad s = -3, ..., 0, \qquad t = 1, 2, ..., N \tag{21}$$

$$u_{4t+s} = \phi(L)\epsilon_{4t+s}, \epsilon_{4t+s} \sim NIID(0, 1) \tag{22}$$

with $\epsilon_{4t+s} = 0$ for $t \leq 0$ and $c$ is the local-to unity parameter. For investigation of the finite sample size performance, we pick $c = 0$ and for the finite sample power performance, we pick $c \in \{7, 13.5\}$

By expanding $\phi(L) = \theta(L)\psi(L)^{-1}$ with two finite lag operators $\theta(L)$ and $\psi(L)$, we consider 5 different types of serial correlation scenarios for $u_{4t+s}$ as follows:

$$(1 - \psi_1 L - \psi_2 L^2 - \psi_4 L^4)u_{4t+s} = (1 + \theta_1 L + \theta_2 L^2 + \theta_4 L^4)\epsilon_{4t+s} \tag{23}$$

where $\epsilon_{4t+s}$ is a martingale difference sequence. For the Monte Carlo simulations, in addition to no serial correlation case, we consider the parametrisation in equation (23) as follows: **(i)** $\theta_1 = 0.5$, $\theta_2 = \theta_4 = 0$, $\psi_1 = \psi_2 = \psi_4 = 0$, **(ii)** $\psi_1 = 0.5$, $\theta_1 = \theta_2 = \theta_4 = 0$, $\psi_2 = \phi_4 = 0$, **(iii)** $\psi_1 = -0.5$, $\theta_1 = 0.5$, $\theta_2 = \theta_4 = 0$, $\psi_2 = \psi_4 = 0$, **(iv)** $\psi_2 = -0.5$, $\theta_1 = \theta_2 = \theta_4 = 0$, $\psi_1 = \psi_4 = 0$, and **(v)** $\theta_4 = 0.5$, $\psi_4 = 0.5$, $\theta_1 = \theta_2 = 0$, $\psi_1 = \psi_2 = 0$.

Results are reported for the $t_0$, $t_{S/2}$, $F_1$, $F_{0,\lfloor S/2 \rfloor}$ and $F_{1,\lfloor S/2 \rfloor}$ statistics of the HEGY tests, $\tau_0(d_1)$, $\tau_{S/2}(d_1)$, $\tilde{\tau}_1(d_1)$, $\tau_{0,\lfloor S/2 \rfloor}(d_1)$ and $\tau_{1,\lfloor S/2 \rfloor}(d_1)$ of the fractional seasonal variance ratio tests and $\tau_0^{**}(d_1)$, $\tau_{S/2}^{**}(d_1)$, $\tilde{\tau}_1^{**}(d_1)$, $\tau_{0,\lfloor S/2 \rfloor}^{**}(d_1)$ and $\tau_{1,\lfloor S/2 \rfloor}^{**}(d_1)$ of the wavelet bootstrap fractional seasonal variance ratio tests.

In our proposed tests, for the investigation of impact of the fractional integration parameter $d_1$ in our test statistics, $d_1$ is chosen as $d_1 = 0.1$, $d_1 = 0.5$ and $d_1 = 1$.[9]. The data is generated from (21)-(22) with the sample sizes of $N = 100$ and $N = 400$. Additionally, 0.05 nominal significance level is used.

We have six tables for the simulation results. Tables 2, 3 and 4 consist of the finite sample size and size adjusted power of the standard HEGY tests, the FSVR and the bootstrap counterparts at the zero and Nyquist frequencies with no de-trending, de-meaning of seasonal intercepts and de-trending of seasonal intercepts and seasonal trends, respectively.

[9]The results with other values of $d_1$ are available upon request

In Table 5, 6 and 7, the corresponding results are reported for the harmonic frequency and the joint frequencies.

### *6.1. The Finite Sample Size Performance*

For the investigation of the size performances of the tests, we take $c = 0$ in (21), which corresponds to the null hypothesis of seasonal nonstationarity. The reported results are the empirical rejection frequencies of the proposed tests based on the corresponding 0.05 level asymptotic critical values.

Table 2 shows that, under no de-trending and no serial correlation, at the zero and Nyquist frequencies, our proposed tests perform very well. Additionally, our wavestrapped tests display no size distortion. The size performance of our tests are very stable at all serial correlation scenarios and at all different parameter values of $d_1$. Further, HEGY tests are also exhibiting satisfactory performance. However, there is no single best test under these scenarios.

Table 3 and 4 shows that under demeaning and detrending, our tests perform well and there are not significant size distortions in both standard FSVR tests and the wavestrapped counterparts. Under scenario (v), when the sample size is $N = 400$, the HEGY tests at the zero and Nyquist frequencies experiences over-sizing around 0.08. Under this scenario, the FSVR tests and their wavelet bootsrap counterparts have size around 0.04.

At the harmonic and the joint frequencies, it can be seen from 5, the FSVR tests and bootstrapped counterparts have desirable properties which are near to the nominal significance level. Under demeaning and detrending, at some cases, we observed mild under-sizing for our tests.

### *6.2. The Finite Sample Size Adjusted Power Performance*

For the finite sample size adjusted power comparison, we consider the near-seasonally integrated alternatives of the tests by taking we $c \in \{7, 13.5\}$ in (21) for different frequencies. In general, it is seen that for our tests, the size adjusted power is maximized at the parameter $d_1 = 0.1$ at all frequencies. This result is consistent with the findings of Nielsen (2009). Therefore, hereafter, the results are reported based on this value of the parameter $d_1$.

First, in Table 2, under no de-trending, the size adjusted power performance of the HEGY tests are higher than our proposed tests. The important result that can be deduced from the results in Table 2, our tests do not have significant power loss. Moreover, for example when $N = 100$, under the serial correlation scenario $(v)$, as AR and MA order increases, the finite sample power of the HEGY tests decrease sharply. In this case, the HEGY tests at the zero and Nyquist frequencies can not beat the performance of the FSVR tests.

Table 3 and Table 4 demonstrate the finite sample size adjusted power results of all tests at the zero and Nyquist frequencies under demeaning and detrending, respectively. These tables show that the proposed tests can compete with the HEGY tests. To begin with, it can be seen that there is a significant power loss in HEGY tests when OLS demeaning and detrending procedures are applied. For example, at the Nyquist frequency with the sample size N=100, even under no serial correlation, the size adjusted power is 0.93 under no detrending, but it is reduced to 0.54 and 0.31 under demeaning and detrending. Under the same case, our results are 0.72, 0.57 and 0.31, respectively. Our competitive size adjusted power increases when there is serial correlation in the innovations. As the MA and AR order increases, the size-adjusted powers experience sharp declines in both frequencies. For example, under scenario $(v)$, when $N = 100$, the size adjusted power of the HEGY tests at the zero and Nyquist frequencies, decrease to 0.20 and 0.18 which are less than our tests for all $d_1$ parameters.

Consider the size adjusted power results for the tests at the harmonic and joint frequencies. Table 4 illustrates that, under no detrending, our tests perform as good as the HEGY tests at these frequencies. In Table 5 and Table 6, the results under demeaning and detrending are reported. It can be clearly seen that our results are better than the corresponding HEGY tests at all given frequencies. This indicates that our nonparametric tests enjoy better performance at joint testing than the HEGY type tests.

## 7. Conclusion

This paper brings some technical innovations to the seasonal unit root testing literature. First, we have developed a new class of non-parametric seasonal unit root tests by making a seasonal generalization to Nielsen (2009). The proposed seasonal test statistics are constructed as a ratio of the sample variance of the seasonally transformed series and that of a fractional partial sum of the same transformed series. The test statistics at all frequencies are free of tuning parameters, such as lag length and bandwith etc. Further, we extend the seasonal variance ratio framework by proposing joint frequency tests. This type of analysis is missing in Taylor (2005).

Second important contribution is on fractional integration literature. We introduce a new fractionally transformed seasonal series, $\tilde{x}_{St+s}$. Although these series are univariate, we utilize the theory for type II multivariate fractional Brownian motions so as to find the asymptotic distribution for these objects. Further, one can assume the fractionally transformed seasonal series as data generation process and conduct test on the fractional integration parameter. We leave this issue for a future study.

Third, we have proposed a wavelet based bootstrap implementation for the constructed nonparametric seasonal unit root tests. With the help of this new technique, we have directly used seasonally differenced data for re-sampling without estimating any parameters. Thanks to the wavestrapping of the FSVR tests, the finite sample size performance with the different serial correlation structures have desirable results.

Last but not the least, in the Monte Carlo simulations, we investigate the effect of the fractional integration parameter $d_1$ in our testing procedure. The best results for both size and size adjusted power exercises are obtained when $d_1 = 0.1$. This result is consistent with the findings of Nielsen (2009) for standard unit root tests. Moreover, these simulations reveal that our proposed tests are not exposed to severe size distortions under any scenario. Especially under demeaning and detrending, the size adjusted power performance of our tests is quite satisfactory compared to the HEGY tests.

**Appendix A: Proofs**

Throughout the proofs, we will utilize the circulant matrices. These matrices can be defined as in del Barrio Castro et al. (2012):

$$\begin{aligned}
C_0 = v_0'v_0 &= \mathbf{Circ}[1, 1, 1, ..., 1] \\
C_{S/2} = v_{S/2}'v_{S/2} &= \mathbf{Circ}[1, -1, 1, ..., -1, 1] \\
C_i = v_i'v_i &= \mathbf{Circ}[cos(0), cos(\omega_i, cos(2\omega_i)), ..., cos\left((S-1)\omega_i\right)] \\
C_i^* = v_i^{*'}v_i^* &= \mathbf{Circ}[sin(0), sin(\omega_i, sin(2\omega_i)), ..., sin\left((S-1)\omega_i\right)]
\end{aligned}$$

with

$$\begin{aligned}
v_0 &= \begin{bmatrix} 1, 1, ..., 1 \end{bmatrix} \\
v_{S/2} &= \begin{bmatrix} -1, 1, -1, ..., 1 \end{bmatrix} \\
v_i &= \begin{bmatrix} cos(w_j[1-S]) & cos(w_j[2-S]) & ... & cos(0) \\ sin(w_j[1-S]) & sin(w_j[2-S]) & ... & sin(0) \end{bmatrix} \\
v_i^* &= \begin{bmatrix} -sin(w_j[1-S]) & -sin(w_j[2-S]) & ... & -sin(0) \\ cos(w_j[1-S]) & cos(w_j[2-S]) & ... & cos(0) \end{bmatrix}
\end{aligned}$$

where $v$s are the associated row vectors and matrices that create the circulant matrices. These matrices above have the following properties:

$$\begin{aligned}
C_0\Psi(1) &= \psi(1)C_0 \\
C_{S/2}\Psi(1) &= \psi(-1)C_{S/2} \\
C_j\Psi(1) &= b_jC_j + a_jC_j^* \ for\ all\ j = 1, ..., S^* \\
C_j^*\Psi(1) &= b_jC_j^* - a_jC_j \ for\ all\ j = 1, ..., S^*
\end{aligned}$$

On the other hand, the interaction of the circulant matrices with each other engender interesting results:

$$\begin{aligned} C_0 C_0 &= SC_0 \\ C_{S/2} C_{S/2} &= SC_{S/2} \\ C_j C_j &= (S/2) C_j \ \textit{for all}\ j = 1, ..., S^* \\ C_j^* C_j^* &= (S/2) C_j \ \textit{for all}\ j = 1, ..., S^* \\ C_j C_j^* &= C_j^* C_j = (S/2) C_j^* \ \textit{for all}\ j = 1, ..., S^* \end{aligned}$$

with other cross products are all zero matrices.

**Proof of Lemma 1**. The proof of this lemma can be found in Lemma 1 of Castro, Osborn and Taylor (2010). □

**Proof of Lemma 2**. The proof follows from the same argument in proof of Lemma 6.(d) of Nielsen (2010). Since the operator $\Delta_+^{-d}$ satisfies $\Delta_+^{-d_1}\Delta_+^{-d_2}Y_t = \Delta_+^{-d_2}\Delta_+^{-d_1}Y_t = \Delta_+^{-d_1-d_2}Y_t$, we can write

$$\tilde{X}_r = \Delta_+^{-d_1}\Delta_+^{-1}E_r = \Delta_+^{-1}\Delta_+^{-d_1}E_r = \sum_{i=1}^{r} \Delta_+^{-d_1}E_r$$

Now define $V_r = \Delta_+^{-d_1}E_r$ which satisfies the conditions of Theorem 1 in Marinucci and Robinson (2000), we obtain :

$$N^{1/2-d_1} \sum_{i=1}^{\lfloor rN \rfloor} V_i \xrightarrow{D} \sigma\Psi(1)W_{d_1+1}(r)$$

□

**Proof of Theorem 1**. Consider numerator of $\tau_j(d_1)$ for $j = 0, S/2$, this object is given in Castro, Osborn and Taylor (2010) as:

$$\begin{aligned} T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}^2 &= T^{-2}\sum_{t=1}^{N} SX_t'C_jX_t + o_p(1) \quad for \quad j = 0, S/2 \\ &\xrightarrow{D} \frac{1}{S}\int_0^1 W(r)'\Psi(1)'C_j\Psi(1)W(r)dr \\ &= \begin{cases} \frac{\sigma^2\psi(1)^2}{S}\int\limits_0^1 W(r)'C_0W(r)dr \ \textit{if}\ j = 0 \\ \frac{\sigma^2\psi(-1)^2}{S}\int\limits_0^1 W(r)'C_{S/2}W(r)dr \ \textit{if}\ j = S/2 \end{cases} \end{aligned} \tag{24}$$

The first equality stems from the properties of circulant matrices $C_0$ and $C_{S/2}$. The weak convergence is a result of Lemma 1 and CMT. But also note that the term $\frac{1}{S}$ appears since the partial sum convergence is over $N$ not $T = NS$, thus we have an extra term $\frac{1}{S^2}$. The final convergences differ for $j = 0, S/2$ because of the circulant matrices again.

Similarly, for $j = 1, ..., S^*$ Castro, Osborn and Taylor (2010) shows that:

$$T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}^2 = T^{-2}\sum_{t=1}^{N}\frac{S}{2}X_t' C_j X_t + o_p(1) \quad \textbf{for} \quad j = 1, ..., S^*$$
$$\xrightarrow{D} \frac{1}{2S}\int_0^1 W(r)'\Psi(1)'C_j\Psi(1)W(r)dr$$
$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int_0^1 W(r)'C_jW(r)dr \ \ if\ j = 1, ..., S^* \tag{25}$$

and

$$T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}^{*2} = T^{-2}\sum_{t=1}^{N}\frac{S}{2}X_t' C_j^* X_t + o_p(1) \quad \textbf{for} \quad j = 1, ..., S^*$$
$$\xrightarrow{D} \frac{1}{2S}\int_0^1 W(r)'\Psi(1)'C_j^*\Psi(1)W(r)dr$$
$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int_0^1 W(r)'C_j^*W(r)dr \ \ if\ j = 1, ..., S^* \tag{26}$$

Remark that in Castro, Osborn and Taylor (2010), it is shown that the objects in (25) and (26) are the same. Now, we can focus on the denominators and apply the same procedure:

$$T^{-2-2d_1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} \tilde{x}_{j,St+s}^2 = T^{-2-2d_1}\sum_{t=1}^{N} S\tilde{X}_t' C_j \tilde{X}_t + o_p(1) \quad for \quad j = 0, S/2$$
$$\xrightarrow{D} \frac{1}{S}\int_0^1 W_{d_1+1}(r)'\Psi(1)'C_j\Psi(1)W_{d_1+1}(r)dr$$
$$= \begin{cases} \frac{\sigma^2\psi(1)^2}{S}\int_0^1 W_{d_1+1}(r)'C_0W_{d_1+1}(r)dr \ \ if\ j = 0 \\ \frac{\sigma^2\psi(-1)^2}{S}\int_0^1 W_{d_1+1}(r)'C_{S/2}W_{d_1+1}(r)dr \ \ if\ j = S/2 \end{cases} \tag{27}$$

The week convergence stem from Lemma 2 and the rest can derived from the above results. Finally, we can also demonstrate,

$$T^{-2-2d_1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} \tilde{x}_{j,St+s}^2 = T^{-2-2d_1}\sum_{t=1}^{N}\frac{S}{2}\tilde{X}_t' C_j \tilde{X}_t + o_p(1) \quad \textbf{for} \quad j = 1, ..., S^*$$
$$\xrightarrow{D} \frac{1}{2S}\int_0^1 W_{d_1+1}(r)'\Psi(1)'C_j\Psi(1)W_{d_1+1}(r)dr$$
$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int_0^1 W_{d_1+1}(r)'C_jW_{d_1+1}(r)dr \ \ if\ j = 1, ..., S^* \tag{28}$$

$$T^{-2-2d_1}\sum_{t=1}^{N}\sum_{s=1-S}^{0}\tilde{x}_{j,St+s}^{*2} = T^{-2-2d_1}\sum_{t=1}^{N}\frac{S}{2}\tilde{X}_t' C_j^* \tilde{X}_t + o_p(1) \quad \textbf{for} \quad j=1,...,S^*$$

$$\xrightarrow{D} \frac{1}{2S}\int_0^1 W_{d_1+1}(r)'\Psi(1)'C_j\Psi(1)W_{d_1+1}(r)dr$$

$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int_0^1 W_{d_1+1}(r)'C_j^* W_{d_1+1}(r)dr \ \textit{if}\ j=1,...,S^* \quad (29)$$

where the asymptotic objects in (28) and (29) are same from the similar arguments made for numerator counterparts. After obtaining these fundamental objects, for all tests $\tau_0(d_1)$, $\tau_{S/2}(d_1)$, $\tau_j(d_1)$, $\tau_j^*(d_1)$, $F_j(d_1)$, $F_{1,...,S/2}(d_1)$ and $F_{0,...,S/2}(d_1)$ the results follow by invoking CMT. Also note that the scales for the numerator and denominator of these are the same. Thus, the division cancels the impact of the serial correlation and other complicated dynamics appearing in the asymptotic objects.

Further, following del Barrio Castro et al. (2012), we can write $W(s)'C_jW(s) = W(s)'v_j'v_jW(s)$ for $j=0,S/2$ and $W(s)'C_iW(s) = W(s)'v_i'v_iW(s)$. However, $W(s)'v_j'$ is just linear combination of independent Brownian motions. For $i=1,...,S^*$, $W(s)'v_i$ is two dimensional Brownian motion. Instead, we can define two new linear combination of independent Brownian motions as $W(s)'c_i$ and $W(s)'c_i^*$ where $c_i$ and $c_i^*$ are the first rows of $v_j$ and $v_j^*$ respectively, for all $j=1,...,S^*$. Moreover, we can make the same arguments for the Fractional Brownian motions appearing in the denominators which is clear from equation (19) for all cases.

□

### Appendix B: Wavelet Filters

Wavelets are simply wave like functions. Unlike other wave like sinusoidal functions, they oscillate in a finite domain and die out. We can use these functions to build filters which can extract the different frequency components of the observed data. We first construct high pass, and low pass wavelet filters from wavelet functions. We define $h = (h_0, h_1, ..., h_{L-1})$ as the high pass filter with filter length $L$. Since this filter is constructed from wavelets it needs to satisfy the following conditions:

$$\sum_{l=0}^{L-1} h_l = 0 \quad \sum_{l=0}^{L-1} h_l^2 = 1$$

These equations indicate that the filter fluctuates around 0 and has unit energy. Every deviation from zero is canceled out by the movement in the opposite direction, but after some time these deviations are neutralized. Furthermore, we can obtain the complementary (low pass filter) filter $g = (g_0, g_1, ..., g_{L-1})$ by using the quadrature mirror relation:

$$g_l = (-1)^l + 1h_{L-1-l} \ \textit{for}\ l = 0,1,...,L-1$$

Using the high pass $h$ and the low pass $g$ filters with Pyramid algorithm, we can obtain the Discrete wavelet transform[10]. Let $z_t$ be a time series with dyadic length of $2^J$, then the pyramid algorithm for $l$ level wavelet transform is as following:

1. Apply the high pass and the low pass filters to the observed series via convolution:

$$V_{1,t} = \sum_{i=0}^{L-1} g_l z_{2t-i\mathrm{Mod}T} \qquad W_{1,t} = \sum_{i=0}^{L-1} h_l z_{2t-i\mathrm{Mod}T}$$

   to obtain first level wavelet coefficients $W_{1,t}$ and the approximation coefficients $V_{1,t}$

2. For obtaining the further level coefficients, apply the high pass and low pas filters to the previous level approximation coefficients only:

$$V_{j,t} = \sum_{i=0}^{L-1} g_l V_{j-1,2t-i\mathrm{Mod}T} \qquad W_{j,t} = \sum_{i=0}^{L-1} h_l V_{j-1,2t-i\mathrm{Mod}T} \qquad \textit{for all } j = 2, ..., l$$

   Remark that $l$ should be smaller than $J$, and at each level we lose the half of the previous level sample size. This leads that at each level $V_j$ is associated with the lower frequencies. Accordingly, while $V_l$ is the lowers frequency coefficient, $W_1$ is the highest frequency coefficient obtain from this transformation. Further, in general $W_j$s are considered as the high frequency coefficients (they are generated by the high pass filter) and they are asymptotically uncorrelated.

   In our analysis, we apply this transformation to a matrix data. We simply use the algorithm above for each column of the observation matrix. Another important feature of wavelet transformation is that we can perfectly reconstruct the observed series from the filtered series. This can be done by using the reconstruction filter with upsampling.

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[10]There are other types of wavelet transforms such as the Maximum Overlap Discrete Wavelet Transform, the Continous Wavelet transforms, the Discrete Wavelet Packet Transform,..., etc. But the DWT suits best for our analysis.

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# Tables

Table 1: Asymptotic Critical Values for the Fractional Seasonal Variance Ratio Tests for $S = 4$ and $S = 12$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S = 4$ | | | | | | | | | | | | | |
| | | $\tau_0$ | | | $\tilde{\tau}_1$ | | | $\tau_{0S/2}$ | | | $\tau_{1S/2}$ | | |
| | $(1-\gamma)$ | 90% | 95% | 99% | 90% | 95% | 99% | 90% | 95% | 99% | 90% | 95% | 99% |
| $d_1$ | Deterministic | | | | | | | | | | | | |
| 0.1 | 0 | 1.541 | 1.622 | 1.774 | 1.403 | 1.468 | 1.582 | 1.380 | 1.418 | 1.490 | 1.388 | 1.434 | 1.524 |
| 0.1 | 1 | 1.761 | 1.822 | 1.937 | 1.665 | 1.709 | 1.793 | 1.645 | 1.673 | 1.726 | 1.652 | 1.686 | 1.751 |
| 0.1 | 2 | 1.927 | 1.980 | 2.081 | 1.850 | 1.888 | 1.962 | 1.827 | 1.853 | 1.901 | 1.836 | 1.866 | 1.923 |
| 0.5 | 0 | 6.739 | 8.490 | 12.663 | 4.378 | 5.418 | 7.578 | 4.649 | 5.267 | 6.638 | 4.639 | 5.389 | 7.064 |
| 0.5 | 1 | 12.259 | 14.484 | 19.479 | 9.423 | 10.677 | 13.512 | 9.427 | 10.242 | 11.951 | 9.494 | 10.473 | 12.548 |
| 0.5 | 2 | 19.750 | 22.452 | 28.293 | 16.126 | 17.807 | 21.220 | 15.798 | 16.829 | 19.069 | 16.000 | 17.243 | 19.849 |
| 1 | 0 | 33.829 | 49.799 | 102.803 | 15.550 | 22.600 | 41.634 | 21.203 | 27.036 | 44.003 | 20.097 | 26.565 | 44.394 |
| 1 | 1 | 69.254 | 98.448 | 181.725 | 39.293 | 51.687 | 82.181 | 45.813 | 56.057 | 82.014 | 44.629 | 55.956 | 85.556 |
| 1 | 2 | 226.917 | 290.061 | 448.477 | 155.550 | 186.785 | 262.340 | 160.845 | 182.759 | 236.907 | 161.068 | 186.610 | 249.927 |
| $S = 12$ | | | | | | | | | | | | | |
| 0.1 | 0 | 1.542 | 1.624 | 1.776 | 1.402 | 1.467 | 1.584 | 1.314 | 1.334 | 1.373 | 1.312 | 1.333 | 1.374 |
| 0.1 | 1 | 1.761 | 1.824 | 1.941 | 1.665 | 1.711 | 1.797 | 1.595 | 1.610 | 1.640 | 1.594 | 1.610 | 1.641 |
| 0.1 | 2 | 1.927 | 1.981 | 2.082 | 1.850 | 1.889 | 1.962 | 1.784 | 1.797 | 1.824 | 1.784 | 1.798 | 1.827 |
| 0.5 | 0 | 6.749 | 8.484 | 12.539 | 4.388 | 5.424 | 7.564 | 3.717 | 4.000 | 4.602 | 3.673 | 3.964 | 4.599 |
| 0.5 | 1 | 12.243 | 14.464 | 19.466 | 9.390 | 10.596 | 13.376 | 8.110 | 8.498 | 9.316 | 8.066 | 8.468 | 9.318 |
| 0.5 | 2 | 19.657 | 22.326 | 28.271 | 16.193 | 17.814 | 21.237 | 14.096 | 14.620 | 15.661 | 14.060 | 14.603 | 15.708 |
| 1 | 0 | 33.529 | 49.350 | 100.270 | 15.539 | 22.619 | 42.246 | 14.427 | 16.818 | 22.637 | 13.874 | 16.226 | 21.882 |
| 1 | 1 | 69.907 | 98.257 | 179.068 | 39.519 | 51.669 | 83.103 | 33.087 | 37.147 | 46.633 | 32.083 | 36.246 | 45.583 |
| 1 | 2 | 227.078 | 290.413 | 457.300 | 155.374 | 186.260 | 264.559 | 129.095 | 139.048 | 160.268 | 127.472 | 137.998 | 159.390 |

Note: We drop the $(d_1)$ notation. We use 100000 Monte Carlo simulations to obtain the critical values with N=1000. $\gamma$ denotes the significance level. Deterministic=0 is for the case of no deterministic adjustment, Deterministic=1 denotes the case of only seasonal intercept adjustment and Deterministic=2 denotes the case of both seasonaln intercept and trend adjustment.

Table 2: Finite Sample Size and Size-adjusted Power for One Sided Fractional Seasonal Variance Ratio Tests and HEGY Tests without Demeaning or Detrending

| | | | $N = 100$ | | | | | | $N = 400$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $d_1$ | $c$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}$ | $\tau_{S/2}^{**}$ | $t_{S/2}$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}$ | $\tau_{S/2}^{**}$ | $t_{S/2}$ |
| | | 0 | 0.05 | 0.05 | 0.06 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 |
| | 0.1 | 7 | 0.35 | 0.36 | 0.51 | 0.36 | 0.37 | 0.52 | 0.35 | 0.35 | 0.51 | 0.36 | 0.34 | 0.51 |
| | | 13.5 | 0.70 | 0.71 | 0.92 | 0.72 | 0.73 | 0.93 | 0.68 | 0.68 | 0.93 | 0.70 | 0.69 | 0.94 |
| | | 0 | 0.05 | 0.05 | 0.04 | 0.06 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 |
| i.i.d | 0.5 | 7 | 0.27 | 0.28 | 0.52 | 0.25 | 0.26 | 0.52 | 0.26 | 0.26 | 0.51 | 0.24 | 0.26 | 0.48 |
| | | 13.5 | 0.51 | 0.52 | 0.93 | 0.49 | 0.5 | 0.93 | 0.49 | 0.48 | 0.94 | 0.48 | 0.51 | 0.93 |
| | | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 1 | 7 | 0.18 | 0.19 | 0.55 | 0.18 | 0.18 | 0.51 | 0.18 | 0.18 | 0.52 | 0.18 | 0.18 | 0.52 |
| | | 13.5 | 0.33 | 0.34 | 0.94 | 0.33 | 0.34 | 0.92 | 0.33 | 0.32 | 0.94 | 0.31 | 0.31 | 0.94 |
| | | 0 | 0.04 | 0.05 | 0.04 | 0.06 | 0.04 | 0.06 | 0.05 | 0.05 | 0.05 | 0.06 | 0.04 | 0.05 |
| | 0.1 | 7 | 0.36 | 0.37 | 0.51 | 0.40 | 0.32 | 0.49 | 0.33 | 0.33 | 0.48 | 0.37 | 0.32 | 0.51 |
| | | 13.5 | 0.71 | 0.72 | 0.90 | 0.78 | 0.71 | 0.90 | 0.67 | 0.67 | 0.92 | 0.73 | 0.67 | 0.93 |
| | | 0 | 0.05 | 0.06 | 0.04 | 0.05 | 0.04 | 0.07 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.06 |
| MA(1): $\theta_1 = 0.5$ | 0.5 | 7 | 0.25 | 0.28 | 0.49 | 0.28 | 0.26 | 0.52 | 0.26 | 0.27 | 0.51 | 0.26 | 0.25 | 0.48 |
| | | 13.5 | 0.49 | 0.52 | 0.89 | 0.56 | 0.53 | 0.91 | 0.49 | 0.5 | 0.93 | 0.51 | 0.49 | 0.92 |
| | | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.04 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 1 | 7 | 0.19 | 0.20 | 0.5 | 0.20 | 0.18 | 0.48 | 0.18 | 0.18 | 0.49 | 0.19 | 0.18 | 0.52 |
| | | 13.5 | 0.33 | 0.35 | 0.90 | 0.36 | 0.34 | 0.89 | 0.32 | 0.33 | 0.92 | 0.33 | 0.32 | 0.94 |
| | | 0 | 0.04 | 0.06 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 0.1 | 7 | 0.33 | 0.36 | 0.47 | 0.37 | 0.34 | 0.53 | 0.36 | 0.37 | 0.49 | 0.35 | 0.33 | 0.51 |
| | | 13.5 | 0.68 | 0.71 | 0.89 | 0.75 | 0.72 | 0.93 | 0.70 | 0.71 | 0.92 | 0.69 | 0.67 | 0.94 |
| | | 0 | 0.04 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 |
| AR(1): $\psi_1 = 0.5$ | 0.5 | 7 | 0.24 | 0.27 | 0.48 | 0.27 | 0.27 | 0.51 | 0.25 | 0.26 | 0.49 | 0.25 | 0.26 | 0.49 |
| | | 13.5 | 0.48 | 0.51 | 0.88 | 0.53 | 0.53 | 0.92 | 0.48 | 0.5 | 0.92 | 0.48 | 0.49 | 0.93 |
| | | 0 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 1 | 7 | 0.19 | 0.19 | 0.49 | 0.19 | 0.19 | 0.5 | 0.17 | 0.19 | 0.5 | 0.18 | 0.18 | 0.52 |
| | | 13.5 | 0.33 | 0.33 | 0.89 | 0.35 | 0.35 | 0.92 | 0.30 | 0.32 | 0.92 | 0.32 | 0.32 | 0.94 |
| | | 0 | 0.05 | 0.06 | 0.05 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 0.1 | 7 | 0.32 | 0.34 | 0.48 | 0.37 | 0.38 | 0.52 | 0.35 | 0.34 | 0.49 | 0.33 | 0.34 | 0.49 |
| | | 13.5 | 0.68 | 0.70 | 0.92 | 0.73 | 0.73 | 0.92 | 0.69 | 0.69 | 0.93 | 0.68 | 0.68 | 0.93 |
| AR(1): $\psi_1 - 0.5$ | | 0 | 0.06 | 0.06 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 0.5 | 7 | 0.24 | 0.26 | 0.52 | 0.27 | 0.28 | 0.53 | 0.25 | 0.26 | 0.52 | 0.24 | 0.26 | 0.48 |
| MA(1): $\theta_1 = 0.5$ | | 13.5 | 0.49 | 0.5 | 0.93 | 0.5 | 0.52 | 0.93 | 0.48 | 0.49 | 0.94 | 0.48 | 0.49 | 0.93 |
| | | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 |
| | 1 | 7 | 0.19 | 0.20 | 0.53 | 0.19 | 0.19 | 0.52 | 0.18 | 0.19 | 0.5 | 0.18 | 0.18 | 0.52 |
| | | 13.5 | 0.33 | 0.35 | 0.93 | 0.34 | 0.35 | 0.94 | 0.31 | 0.32 | 0.94 | 0.32 | 0.32 | 0.94 |
| | | 0 | 0.05 | 0.04 | 0.05 | 0.06 | 0.04 | 0.05 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 |
| | 0.1 | 7 | 0.38 | 0.33 | 0.53 | 0.36 | 0.33 | 0.49 | 0.36 | 0.33 | 0.51 | 0.36 | 0.33 | 0.53 |
| | | 13.5 | 0.76 | 0.71 | 0.93 | 0.75 | 0.71 | 0.91 | 0.71 | 0.68 | 0.94 | 0.71 | 0.68 | 0.94 |
| | | 0 | 0.06 | 0.05 | 0.05 | 0.06 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 |
| AR(2): $\psi_2 = -0.5$ | 0.5 | 7 | 0.26 | 0.25 | 0.51 | 0.28 | 0.25 | 0.5 | 0.26 | 0.27 | 0.51 | 0.27 | 0.25 | 0.49 |
| | | 13.5 | 0.52 | 0.51 | 0.92 | 0.54 | 0.51 | 0.92 | 0.5 | 0.5 | 0.93 | 0.5 | 0.47 | 0.93 |
| | | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | 1 | 7 | 0.19 | 0.19 | 0.51 | 0.19 | 0.18 | 0.51 | 0.20 | 0.19 | 0.5 | 0.17 | 0.17 | 0.52 |
| | | 13.5 | 0.35 | 0.34 | 0.92 | 0.35 | 0.35 | 0.93 | 0.35 | 0.34 | 0.93 | 0.31 | 0.30 | 0.94 |
| | | 0 | 0.03 | 0.06 | 0.05 | 0.03 | 0.06 | 0.05 | 0.04 | 0.06 | 0.06 | 0.04 | 0.06 | 0.07 |
| | 0.11 | 7 | 0.29 | 0.35 | 0.30 | 0.30 | 0.33 | 0.31 | 0.32 | 0.36 | 0.47 | 0.35 | 0.39 | 0.48 |
| | | 13.5 | 0.57 | 0.64 | 0.62 | 0.60 | 0.64 | 0.64 | 0.63 | 0.68 | 0.88 | 0.66 | 0.71 | 0.88 |
| AR(4): $\psi_4 = 0.5$ | | 0 | 0.04 | 0.06 | 0.05 | 0.04 | 0.06 | 0.05 | 0.04 | 0.06 | 0.06 | 0.04 | 0.05 | 0.06 |
| | 0.5 | 7 | 0.24 | 0.27 | 0.30 | 0.23 | 0.26 | 0.31 | 0.25 | 0.27 | 0.48 | 0.26 | 0.27 | 0.47 |
| MA(4): $\theta_4 = 0.5$ | | 13.5 | 0.44 | 0.47 | 0.63 | 0.42 | 0.46 | 0.64 | 0.46 | 0.48 | 0.88 | 0.47 | 0.49 | 0.88 |
| | | 0 | 0.04 | 0.06 | 0.04 | 0.04 | 0.06 | 0.04 | 0.05 | 0.05 | 0.06 | 0.05 | 0.06 | 0.07 |
| | 1 | 7 | 0.16 | 0.17 | 0.33 | 0.16 | 0.19 | 0.33 | 0.17 | 0.19 | 0.48 | 0.17 | 0.19 | 0.45 |
| | | 13.5 | 0.28 | 0.30 | 0.66 | 0.28 | 0.31 | 0.66 | 0.30 | 0.32 | 0.88 | 0.30 | 0.32 | 0.87 |

Note: We drop the $(d_1)$ notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.

Table 3: Finite Sample Size and Size-adjusted Power for One Sided Fractional Seasonal Variance Ratio Tests and HEGY Tests with Demeaning

| | | | $N = 100$ | | | | | | $N = 400$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $d_1$ | $c$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}$ | $\tau_{S/2}^{**}$ | $t_{S/2}$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}($ | $\tau_{S/2}^{**}$ | $t_{S/2}$ |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 0.1 | 7 | 0.24 | 0.17 | 0.18 | 0.25 | 0.17 | 0.18 | 0.24 | 0.16 | 0.17 | 0.24 | 0.16 | 0.18 |
| | | 13.5 | 0.55 | 0.43 | 0.53 | 0.57 | 0.45 | 0.54 | 0.53 | 0.40 | 0.5 | 0.53 | 0.41 | 0.51 |
| | | 0 | 0.04 | 0.04 | 0.05 | 0.04 | 0.03 | 0.04 | 0.06 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| i.i.d | 0.5 | 7 | 0.23 | 0.18 | 0.19 | 0.24 | 0.17 | 0.19 | 0.22 | 0.15 | 0.18 | 0.23 | 0.17 | 0.17 |
| | | 13.5 | 0.5 | 0.41 | 0.54 | 0.5 | 0.41 | 0.56 | 0.44 | 0.34 | 0.51 | 0.47 | 0.37 | 0.51 |
| | | 0 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.04 | 0.05 |
| | 1 | 7 | 0.21 | 0.16 | 0.18 | 0.22 | 0.17 | 0.18 | 0.21 | 0.16 | 0.18 | 0.21 | 0.16 | 0.16 |
| | | 13.5 | 0.40 | 0.33 | 0.55 | 0.40 | 0.34 | 0.53 | 0.39 | 0.31 | 0.53 | 0.37 | 0.30 | 0.49 |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.1 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.07 | 0.02 | 0.06 |
| | 0.1 | 7 | 0.25 | 0.18 | 0.15 | 0.20 | 0.11 | 0.22 | 0.24 | 0.18 | 0.17 | 0.23 | 0.1 | 0.18 |
| | | 13.5 | 0.56 | 0.47 | 0.42 | 0.54 | 0.36 | 0.60 | 0.53 | 0.44 | 0.48 | 0.53 | 0.31 | 0.52 |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.08 | 0.02 | 0.05 | 0.04 | 0.03 | 0.05 | 0.06 | 0.03 | 0.06 |
| MA(1): $\theta_1 = 0.5$ | 0.5 | 7 | 0.23 | 0.17 | 0.15 | 0.22 | 0.13 | 0.24 | 0.23 | 0.17 | 0.17 | 0.24 | 0.14 | 0.20 |
| | | 13.5 | 0.47 | 0.39 | 0.42 | 0.51 | 0.35 | 0.60 | 0.48 | 0.39 | 0.48 | 0.48 | 0.33 | 0.53 |
| | | 0 | 0.04 | 0.04 | 0.04 | 0.07 | 0.03 | 0.05 | 0.05 | 0.04 | 0.05 | 0.06 | 0.03 | 0.05 |
| | 1 | 7 | 0.22 | 0.18 | 0.16 | 0.22 | 0.14 | 0.25 | 0.22 | 0.17 | 0.16 | 0.22 | 0.13 | 0.19 |
| | | 13.5 | 0.40 | 0.34 | 0.45 | 0.43 | 0.30 | 0.62 | 0.38 | 0.32 | 0.47 | 0.40 | 0.28 | 0.53 |
| | | 0 | 0.03 | 0.03 | 0.05 | 0.06 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.25 | 0.18 | 0.15 | 0.23 | 0.13 | 0.20 | 0.24 | 0.17 | 0.17 | 0.23 | 0.12 | 0.19 |
| | | 13.5 | 0.56 | 0.44 | 0.44 | 0.54 | 0.37 | 0.55 | 0.53 | 0.42 | 0.47 | 0.52 | 0.33 | 0.51 |
| | | 0 | 0.04 | 0.04 | 0.05 | 0.06 | 0.03 | 0.04 | 0.05 | 0.04 | 0.05 | 0.06 | 0.03 | 0.05 |
| AR(1): $\psi_1 = 0.5$ | 0.5 | 7 | 0.22 | 0.18 | 0.15 | 0.22 | 0.14 | 0.19 | 0.23 | 0.19 | 0.17 | 0.23 | 0.14 | 0.18 |
| | | 13.5 | 0.47 | 0.40 | 0.44 | 0.49 | 0.36 | 0.55 | 0.45 | 0.40 | 0.48 | 0.47 | 0.33 | 0.50 |
| | | 0 | 0.04 | 0.04 | 0.05 | 0.06 | 0.03 | 0.04 | 0.05 | 0.04 | 0.05 | 0.06 | 0.03 | 0.05 |
| | 1 | 7 | 0.23 | 0.19 | 0.15 | 0.21 | 0.14 | 0.20 | 0.21 | 0.17 | 0.17 | 0.20 | 0.15 | 0.18 |
| | | 13.5 | 0.41 | 0.36 | 0.44 | 0.39 | 0.28 | 0.56 | 0.38 | 0.32 | 0.48 | 0.37 | 0.30 | 0.51 |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 0.1 | 7 | 0.23 | 0.16 | 0.17 | 0.24 | 0.16 | 0.20 | 0.23 | 0.16 | 0.17 | 0.25 | 0.16 | 0.18 |
| | | 13.5 | 0.55 | 0.44 | 0.53 | 0.55 | 0.43 | 0.55 | 0.54 | 0.43 | 0.52 | 0.55 | 0.42 | 0.52 |
| AR(1): $\psi_1 = 0.5$ | | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.04 |
| | 0.5 | 7 | 0.24 | 0.16 | 0.17 | 0.23 | 0.17 | 0.18 | 0.22 | 0.16 | 0.16 | 0.23 | 0.18 | 0.19 |
| MA(1): $\theta_1 = 0.5$ | | 13.5 | 0.5 | 0.39 | 0.52 | 0.49 | 0.39 | 0.53 | 0.47 | 0.36 | 0.48 | 0.48 | 0.40 | 0.51 |
| | | 0 | 0.05 | 0.04 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 |
| | 1 | 7 | 0.20 | 0.18 | 0.18 | 0.22 | 0.16 | 0.18 | 0.20 | 0.16 | 0.17 | 0.21 | 0.17 | 0.17 |
| | | 13.5 | 0.37 | 0.34 | 0.52 | 0.41 | 0.33 | 0.55 | 0.37 | 0.31 | 0.51 | 0.38 | 0.32 | 0.50 |
| | | 0 | 0.09 | 0.02 | 0.04 | 0.09 | 0.02 | 0.04 | 0.07 | 0.02 | 0.05 | 0.07 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.19 | 0.09 | 0.21 | 0.18 | 0.09 | 0.21 | 0.21 | 0.11 | 0.18 | 0.20 | 0.1 | 0.16 |
| | | 13.5 | 0.49 | 0.28 | 0.57 | 0.47 | 0.29 | 0.56 | 0.5 | 0.31 | 0.52 | 0.5 | 0.31 | 0.49 |
| | | 0 | 0.08 | 0.02 | 0.04 | 0.07 | 0.03 | 0.04 | 0.06 | 0.03 | 0.05 | 0.06 | 0.03 | 0.05 |
| AR(2): $\psi_2 = -0.5$ | 0.5 | 7 | 0.19 | 0.1 | 0.20 | 0.21 | 0.12 | 0.20 | 0.22 | 0.13 | 0.18 | 0.23 | 0.14 | 0.18 |
| | | 13.5 | 0.44 | 0.29 | 0.56 | 0.48 | 0.32 | 0.56 | 0.46 | 0.32 | 0.52 | 0.47 | 0.33 | 0.51 |
| | | 0 | 0.07 | 0.03 | 0.04 | 0.06 | 0.03 | 0.03 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 1 | 7 | 0.20 | 0.14 | 0.20 | 0.21 | 0.13 | 0.20 | 0.20 | 0.15 | 0.19 | 0.20 | 0.15 | 0.18 |
| | | 13.5 | 0.39 | 0.29 | 0.56 | 0.42 | 0.30 | 0.57 | 0.38 | 0.29 | 0.53 | 0.39 | 0.31 | 0.51 |
| | | 0 | 0.01 | 0.03 | 0.05 | 0.01 | 0.03 | 0.05 | 0.03 | 0.04 | 0.07 | 0.03 | 0.04 | 0.07 |
| | 0.1 | 7 | 0.21 | 0.15 | 0.08 | 0.20 | 0.15 | 0.07 | 0.22 | 0.18 | 0.17 | 0.24 | 0.18 | 0.19 |
| | | 13.5 | 0.46 | 0.35 | 0.20 | 0.44 | 0.34 | 0.18 | 0.49 | 0.43 | 0.45 | 0.51 | 0.42 | 0.45 |
| MA(4): $\psi_4 = 0.5$ | | 0 | 0.03 | 0.04 | 0.05 | 0.03 | 0.04 | 0.05 | 0.04 | 0.04 | 0.07 | 0.04 | 0.03 | 0.06 |
| | 0.5 | 7 | 0.19 | 0.16 | 0.07 | 0.22 | 0.17 | 0.07 | 0.22 | 0.17 | 0.17 | 0.23 | 0.18 | 0.18 |
| MA(4): $\theta_4 = 0.5$ | | 13.5 | 0.39 | 0.35 | 0.18 | 0.43 | 0.36 | 0.19 | 0.43 | 0.36 | 0.43 | 0.46 | 0.39 | 0.45 |
| | | 0 | 0.03 | 0.04 | 0.05 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.07 | 0.05 | 0.04 | 0.07 |
| | 1 | 7 | 0.20 | 0.17 | 0.08 | 0.20 | 0.16 | 0.08 | 0.20 | 0.17 | 0.18 | 0.21 | 0.18 | 0.17 |
| | | 13.5 | 0.35 | 0.32 | 0.20 | 0.35 | 0.31 | 0.20 | 0.37 | 0.32 | 0.45 | 0.37 | 0.33 | 0.44 |

Note: We drop the $(d_1)$ notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.

Table 4: Finite Sample Size and Size-adjusted Power for One Sided Fractional Seasonal Variance Ratio Tests and HEGY Tests with Detrending

| | | | $N = 100$ | | | | | | $N = 400$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $d_1$ | $c$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}$ | $\tau_{S/2}^{**}$ | $t_{S/2}$ | $\tau_0$ | $\tau_0^{**}$ | $t_0$ | $\tau_{S/2}$ | $\tau_{S/2}^{**}$ | $t_{S/2}$ |
| i.i.d | 0.1 | 0 | 0.03 | 0.03 | 0.05 | 0.04 | 0.03 | 0.06 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.14 | 0.08 | 0.12 | 0.12 | 0.07 | 0.11 | 0.13 | 0.07 | 0.12 | 0.12 | 0.07 | 0.12 |
| | | 13.5 | 0.34 | 0.23 | 0.33 | 0.31 | 0.21 | 0.31 | 0.32 | 0.20 | 0.30 | 0.32 | 0.19 | 0.32 |
| | 0.5 | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.13 | 0.09 | 0.12 | 0.13 | 0.09 | 0.12 | 0.12 | 0.08 | 0.11 | 0.12 | 0.08 | 0.12 |
| | | 13.5 | 0.30 | 0.23 | 0.34 | 0.30 | 0.23 | 0.33 | 0.29 | 0.20 | 0.30 | 0.28 | 0.20 | 0.31 |
| | 1 | 0 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.12 | 0.08 | 0.12 | 0.12 | 0.08 | 0.12 | 0.13 | 0.08 | 0.11 | 0.12 | 0.08 | 0.12 |
| | | 13.5 | 0.27 | 0.20 | 0.34 | 0.27 | 0.21 | 0.33 | 0.26 | 0.17 | 0.30 | 0.26 | 0.19 | 0.31 |
| MA(1): $\theta_1 = 0.5$ | 0.1 | 0 | 0.03 | 0.03 | 0.05 | 0.1 | 0.02 | 0.06 | 0.05 | 0.03 | 0.05 | 0.08 | 0.01 | 0.06 |
| | | 7 | 0.14 | 0.09 | 0.1 | 0.13 | 0.05 | 0.15 | 0.12 | 0.08 | 0.1 | 0.13 | 0.04 | 0.12 |
| | | 13.5 | 0.32 | 0.23 | 0.26 | 0.34 | 0.17 | 0.40 | 0.31 | 0.22 | 0.28 | 0.32 | 0.12 | 0.30 |
| | 0.5 | 0 | 0.04 | 0.03 | 0.04 | 0.09 | 0.02 | 0.06 | 0.05 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 |
| | | 7 | 0.13 | 0.09 | 0.11 | 0.12 | 0.05 | 0.14 | 0.14 | 0.09 | 0.11 | 0.13 | 0.05 | 0.12 |
| | | 13.5 | 0.31 | 0.24 | 0.27 | 0.31 | 0.16 | 0.38 | 0.28 | 0.21 | 0.28 | 0.29 | 0.14 | 0.31 |
| | 1 | 0 | 0.04 | 0.04 | 0.05 | 0.07 | 0.02 | 0.06 | 0.05 | 0.03 | 0.04 | 0.06 | 0.02 | 0.05 |
| | | 7 | 0.13 | 0.09 | 0.1 | 0.13 | 0.07 | 0.17 | 0.12 | 0.09 | 0.12 | 0.12 | 0.06 | 0.12 |
| | | 13.5 | 0.28 | 0.22 | 0.27 | 0.29 | 0.16 | 0.41 | 0.26 | 0.21 | 0.32 | 0.25 | 0.15 | 0.31 |
| AR(1): $\psi_1 = 0.5$ | 0.1 | 0 | 0.02 | 0.04 | 0.05 | 0.05 | 0.02 | 0.04 | 0.04 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 |
| | | 7 | 0.14 | 0.11 | 0.11 | 0.12 | 0.06 | 0.11 | 0.12 | 0.08 | 0.1 | 0.13 | 0.05 | 0.12 |
| | | 13.5 | 0.33 | 0.27 | 0.26 | 0.33 | 0.17 | 0.34 | 0.29 | 0.22 | 0.27 | 0.30 | 0.14 | 0.30 |
| | 0.5 | 0 | 0.03 | 0.04 | 0.05 | 0.06 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.12 | 0.1 | 0.09 | 0.12 | 0.07 | 0.12 | 0.13 | 0.08 | 0.11 | 0.12 | 0.06 | 0.11 |
| | | 13.5 | 0.29 | 0.25 | 0.26 | 0.30 | 0.19 | 0.35 | 0.28 | 0.21 | 0.29 | 0.28 | 0.17 | 0.31 |
| | 1 | 0 | 0.04 | 0.04 | 0.05 | 0.06 | 0.03 | 0.04 | 0.05 | 0.04 | 0.05 | 0.06 | 0.03 | 0.05 |
| | | 7 | 0.12 | 0.1 | 0.11 | 0.12 | 0.07 | 0.12 | 0.11 | 0.1 | 0.10 | 0.12 | 0.08 | 0.12 |
| | | 13.5 | 0.26 | 0.23 | 0.26 | 0.28 | 0.18 | 0.34 | 0.24 | 0.20 | 0.29 | 0.23 | 0.18 | 0.31 |
| AR(1): $\psi_1 = 0.5$ MA(1): $\theta_1 = 0.5$ | 0.1 | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 |
| | | 7 | 0.13 | 0.08 | 0.12 | 0.12 | 0.07 | 0.11 | 0.13 | 0.07 | 0.12 | 0.13 | 0.07 | 0.12 |
| | | 13.5 | 0.32 | 0.23 | 0.32 | 0.33 | 0.21 | 0.34 | 0.33 | 0.20 | 0.32 | 0.31 | 0.20 | 0.31 |
| | 0.5 | 0 | 0.04 | 0.03 | 0.06 | 0.04 | 0.03 | 0.05 | 0.04 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.13 | 0.08 | 0.12 | 0.13 | 0.08 | 0.11 | 0.13 | 0.08 | 0.12 | 0.13 | 0.08 | 0.11 |
| | | 13.5 | 0.31 | 0.22 | 0.33 | 0.31 | 0.22 | 0.34 | 0.29 | 0.20 | 0.32 | 0.28 | 0.19 | 0.30 |
| | 1 | 0 | 0.05 | 0.04 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 |
| | | 7 | 0.11 | 0.09 | 0.11 | 0.13 | 0.09 | 0.12 | 0.12 | 0.08 | 0.13 | 0.12 | 0.09 | 0.11 |
| | | 13.5 | 0.26 | 0.20 | 0.33 | 0.27 | 0.20 | 0.34 | 0.24 | 0.18 | 0.32 | 0.26 | 0.19 | 0.29 |
| AR(2): $\psi_2 = -0.5$ | 0.1 | 0 | 0.08 | 0.02 | 0.04 | 0.08 | 0.02 | 0.04 | 0.07 | 0.01 | 0.05 | 0.07 | 0.01 | 0.04 |
| | | 7 | 0.12 | 0.04 | 0.13 | 0.12 | 0.04 | 0.13 | 0.12 | 0.04 | 0.11 | 0.13 | 0.04 | 0.11 |
| | | 13.5 | 0.31 | 0.14 | 0.37 | 0.32 | 0.14 | 0.36 | 0.30 | 0.11 | 0.30 | 0.32 | 0.12 | 0.32 |
| | 0.5 | 0 | 0.07 | 0.02 | 0.04 | 0.08 | 0.02 | 0.04 | 0.06 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 |
| | | 7 | 0.13 | 0.06 | 0.13 | 0.12 | 0.06 | 0.12 | 0.12 | 0.06 | 0.12 | 0.12 | 0.06 | 0.12 |
| | | 13.5 | 0.31 | 0.16 | 0.35 | 0.29 | 0.16 | 0.34 | 0.28 | 0.16 | 0.32 | 0.28 | 0.15 | 0.31 |
| | 1 | 0 | 0.07 | 0.02 | 0.04 | 0.06 | 0.02 | 0.04 | 0.06 | 0.02 | 0.05 | 0.06 | 0.03 | 0.05 |
| | | 7 | 0.12 | 0.06 | 0.13 | 0.13 | 0.06 | 0.14 | 0.11 | 0.06 | 0.11 | 0.11 | 0.07 | 0.12 |
| | | 13.5 | 0.27 | 0.16 | 0.35 | 0.28 | 0.17 | 0.38 | 0.24 | 0.16 | 0.31 | 0.23 | 0.17 | 0.31 |
| MA(4): $\psi_4 = 0.5$ MA(4): $\theta_4 = 0.5$ | 0.1 | 0 | 0.01 | 0.03 | 0.04 | 0.01 | 0.03 | 0.03 | 0.02 | 0.03 | 0.08 | 0.02 | 0.03 | 0.08 |
| | | 7 | 0.12 | 0.07 | 0.07 | 0.12 | 0.07 | 0.07 | 0.12 | 0.09 | 0.11 | 0.12 | 0.09 | 0.10 |
| | | 13.5 | 0.27 | 0.17 | 0.14 | 0.26 | 0.17 | 0.14 | 0.30 | 0.23 | 0.26 | 0.28 | 0.22 | 0.23 |
| | 0.5 | 0 | 0.01 | 0.04 | 0.04 | 0.01 | 0.04 | 0.04 | 0.03 | 0.04 | 0.08 | 0.03 | 0.03 | 0.08 |
| | | 7 | 0.12 | 0.09 | 0.07 | 0.12 | 0.09 | 0.08 | 0.12 | 0.1 | 0.11 | 0.12 | 0.09 | 0.10 |
| | | 13.5 | 0.26 | 0.21 | 0.14 | 0.26 | 0.22 | 0.15 | 0.27 | 0.23 | 0.25 | 0.26 | 0.21 | 0.24 |
| | 1 | 0 | 0.02 | 0.04 | 0.04 | 0.02 | 0.04 | 0.04 | 0.04 | 0.04 | 0.08 | 0.04 | 0.04 | 0.07 |
| | | 7 | 0.12 | 0.1 | 0.07 | 0.12 | 0.1 | 0.07 | 0.12 | 0.1 | 0.11 | 0.11 | 0.1 | 0.11 |
| | | 13.5 | 0.24 | 0.21 | 0.14 | 0.23 | 0.20 | 0.14 | 0.25 | 0.21 | 0.25 | 0.25 | 0.22 | 0.25 |

Note: We drop the $(d_1)$ notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.

Table 5: Finite Sample Size and Size-adjusted Power for Joint Fractional Seasonal Variance Ratio Tests and HEGY Tests without Demeaning or Detrending

| | | | $N = 100$ | | | | | | | | | $N = 400$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_1$ | $c$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ |
| i.i.d | 0.1 | 0 | 0.04 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 |
| | | 7 | 0.65 | 0.66 | 0.75 | 0.91 | 0.92 | 0.95 | 0.82 | 0.83 | 0.89 | 0.63 | 0.64 | 0.75 | 0.90 | 0.91 | 0.95 | 0.80 | 0.81 | 0.89 |
| | | 13.5 | 0.95 | 0.96 | 0.99 | 1 | 1 | 1 | 1 | 1 | 1 | 0.95 | 0.95 | 1 | 1 | 1 | 1 | 0.99 | 1 | 1.00 |
| | 0.5 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.06 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 |
| | | 7 | 0.44 | 0.45 | 0.74 | 0.64 | 0.64 | 0.95 | 0.53 | 0.54 | 0.87 | 0.43 | 0.43 | 0.75 | 0.61 | 0.63 | 0.94 | 0.53 | 0.54 | 0.87 |
| | | 13.5 | 0.77 | 0.79 | 0.99 | 0.95 | 0.95 | 1 | 0.88 | 0.89 | 1 | 0.75 | 0.75 | 1 | 0.94 | 0.94 | 1 | 0.87 | 0.87 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 |
| | | 7 | 0.28 | 0.28 | 0.75 | 0.34 | 0.35 | 0.94 | 0.30 | 0.31 | 0.87 | 0.26 | 0.26 | 0.75 | 0.35 | 0.34 | 0.96 | 0.29 | 0.29 | 0.90 |
| | | 13.5 | 0.5 | 0.51 | 0.99 | 0.67 | 0.67 | 1 | 0.58 | 0.59 | 1 | 0.48 | 0.48 | 1 | 0.65 | 0.65 | 1 | 0.56 | 0.56 | 1.00 |
| MA(1): $\theta_1 = 0.5$ | 0.1 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.65 | 0.66 | 0.70 | 0.92 | 0.91 | 0.93 | 0.83 | 0.81 | 0.86 | 0.65 | 0.65 | 0.72 | 0.90 | 0.90 | 0.94 | 0.81 | 0.81 | 0.88 |
| | | 13.5 | 0.95 | 0.96 | 0.99 | 1 | 1 | 1 | 1 | 1 | 1 | 0.95 | 0.95 | 0.99 | 1 | 1 | 1 | 1 | 1 | 1.00 |
| | 0.5 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.43 | 0.44 | 0.70 | 0.65 | 0.64 | 0.93 | 0.54 | 0.52 | 0.85 | 0.40 | 0.41 | 0.70 | 0.62 | 0.62 | 0.95 | 0.52 | 0.51 | 0.86 |
| | | 13.5 | 0.76 | 0.77 | 0.99 | 0.95 | 0.95 | 1 | 0.90 | 0.89 | 1 | 0.73 | 0.74 | 1 | 0.94 | 0.94 | 1 | 0.88 | 0.87 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.26 | 0.27 | 0.69 | 0.36 | 0.35 | 0.93 | 0.30 | 0.29 | 0.86 | 0.26 | 0.26 | 0.74 | 0.33 | 0.32 | 0.96 | 0.29 | 0.29 | 0.89 |
| | | 13.5 | 0.49 | 0.5 | 0.99 | 0.68 | 0.67 | 1 | 0.58 | 0.57 | 1 | 0.48 | 0.47 | 1 | 0.64 | 0.64 | 1 | 0.56 | 0.55 | 1.00 |
| AR(1): $\psi_1 = 0.5$ | 0.1 | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.06 | 0.04 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 |
| | | 7 | 0.65 | 0.63 | 0.72 | 0.90 | 0.90 | 0.95 | 0.82 | 0.79 | 0.88 | 0.63 | 0.64 | 0.75 | 0.90 | 0.91 | 0.95 | 0.80 | 0.80 | 0.88 |
| | | 13.5 | 0.97 | 0.96 | 0.99 | 1 | 1 | 1 | 1 | 1 | 1 | 0.95 | 0.95 | 1 | 1 | 1 | 1 | 0.99 | 0.99 | 1.00 |
| | 0.5 | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.06 | 0.05 | 0.05 | 0.06 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.43 | 0.43 | 0.73 | 0.66 | 0.65 | 0.94 | 0.56 | 0.54 | 0.87 | 0.41 | 0.41 | 0.75 | 0.58 | 0.61 | 0.95 | 0.51 | 0.52 | 0.88 |
| | | 13.5 | 0.78 | 0.77 | 0.99 | 0.95 | 0.95 | 1 | 0.91 | 0.90 | 1 | 0.74 | 0.75 | 1 | 0.93 | 0.94 | 1 | 0.86 | 0.87 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 |
| | | 7 | 0.28 | 0.28 | 0.73 | 0.35 | 0.35 | 0.94 | 0.32 | 0.31 | 0.88 | 0.26 | 0.26 | 0.73 | 0.32 | 0.33 | 0.95 | 0.28 | 0.28 | 0.88 |
| | | 13.5 | 0.53 | 0.54 | 0.99 | 0.68 | 0.68 | 1 | 0.62 | 0.61 | 1 | 0.48 | 0.48 | 1 | 0.63 | 0.64 | 1 | 0.55 | 0.56 | 1.00 |
| AR(1): $\psi_1 = 0.5$ MA(1): $\theta_1 = 0.5$ | 0.1 | 0 | 0.04 | 0.05 | 0.05 | 0.05 | 0.06 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 | 0.06 |
| | | 7 | 0.64 | 0.65 | 0.74 | 0.90 | 0.91 | 0.95 | 0.81 | 0.82 | 0.88 | 0.63 | 0.64 | 0.71 | 0.89 | 0.90 | 0.94 | 0.80 | 0.81 | 0.86 |
| | | 13.5 | 0.95 | 0.96 | 0.99 | 1 | 1 | 1 | 0.99 | 1 | 1 | 0.94 | 0.95 | 1 | 1 | 1 | 1 | 0.99 | 0.99 | 1.00 |
| | 0.5 | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.43 | 0.45 | 0.74 | 0.63 | 0.65 | 0.94 | 0.54 | 0.55 | 0.88 | 0.42 | 0.44 | 0.72 | 0.61 | 0.62 | 0.95 | 0.53 | 0.54 | 0.87 |
| | | 13.5 | 0.77 | 0.78 | 0.99 | 0.94 | 0.95 | 1 | 0.89 | 0.89 | 1 | 0.75 | 0.76 | 1 | 0.94 | 0.94 | 1 | 0.87 | 0.88 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.04 | 0.04 | 0.06 | 0.04 | 0.04 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.28 | 0.27 | 0.75 | 0.35 | 0.37 | 0.95 | 0.30 | 0.31 | 0.88 | 0.27 | 0.27 | 0.75 | 0.34 | 0.36 | 0.95 | 0.30 | 0.31 | 0.89 |
| | | 13.5 | 0.51 | 0.5 | 0.99 | 0.67 | 0.69 | 1 | 0.58 | 0.59 | 1 | 0.48 | 0.48 | 1 | 0.64 | 0.66 | 1 | 0.56 | 0.58 | 1.00 |
| AR(2): $\psi_2 = -0.5$ | 0.1 | 0 | 0.04 | 0.06 | 0.04 | 0.05 | 0.05 | 0.04 | 0.05 | 0.06 | 0.04 | 0.04 | 0.06 | 0.05 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.62 | 0.65 | 0.68 | 0.90 | 0.90 | 0.93 | 0.79 | 0.81 | 0.85 | 0.61 | 0.64 | 0.71 | 0.90 | 0.90 | 0.95 | 0.80 | 0.81 | 0.86 |
| | | 13.5 | 0.93 | 0.94 | 0.98 | 1 | 1 | 1 | 0.99 | 1 | 1 | 0.93 | 0.94 | 0.99 | 1 | 1 | 1 | 0.99 | 0.99 | 1.00 |
| | 0.5 | 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| | | 7 | 0.40 | 0.42 | 0.66 | 0.62 | 0.60 | 0.93 | 0.51 | 0.5 | 0.84 | 0.41 | 0.42 | 0.70 | 0.62 | 0.63 | 0.94 | 0.52 | 0.54 | 0.86 |
| | | 13.5 | 0.72 | 0.74 | 0.98 | 0.94 | 0.94 | 1 | 0.86 | 0.86 | 1 | 0.72 | 0.74 | 0.99 | 0.94 | 0.94 | 1 | 0.86 | 0.87 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 | 0.04 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.04 |
| | | 7 | 0.25 | 0.27 | 0.67 | 0.34 | 0.34 | 0.93 | 0.30 | 0.29 | 0.84 | 0.25 | 0.25 | 0.71 | 0.33 | 0.32 | 0.94 | 0.27 | 0.28 | 0.86 |
| | | 13.5 | 0.47 | 0.49 | 0.98 | 0.68 | 0.68 | 1 | 0.58 | 0.57 | 1 | 0.46 | 0.46 | 1 | 0.65 | 0.63 | 1 | 0.53 | 0.54 | 1.00 |
| MA(4): $\psi_4 = 0.5$ MA(4): $\theta_4 = 0.5$ | 0.1 | 0 | 0.03 | 0.06 | 0.05 | 0.03 | 0.07 | 0.04 | 0.03 | 0.07 | 0.04 | 0.05 | 0.06 | 0.05 | 0.04 | 0.07 | 0.06 | 0.04 | 0.06 | 0.06 |
| | | 7 | 0.55 | 0.60 | 0.48 | 0.83 | 0.87 | 0.74 | 0.72 | 0.78 | 0.62 | 0.59 | 0.62 | 0.75 | 0.88 | 0.90 | 0.95 | 0.77 | 0.81 | 0.88 |
| | | 13.5 | 0.89 | 0.91 | 0.87 | 0.99 | 1 | 0.99 | 0.98 | 0.99 | 0.96 | 0.92 | 0.93 | 0.99 | 1 | 1 | 1 | 0.99 | 0.99 | 1.00 |
| | 0.5 | 0 | 0.04 | 0.06 | 0.05 | 0.03 | 0.06 | 0.05 | 0.04 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 | 0.06 | 0.06 | 0.05 | 0.06 | 0.06 |
| | | 7 | 0.38 | 0.40 | 0.49 | 0.56 | 0.61 | 0.73 | 0.47 | 0.52 | 0.63 | 0.39 | 0.41 | 0.74 | 0.59 | 0.63 | 0.94 | 0.49 | 0.53 | 0.87 |
| | | 13.5 | 0.66 | 0.68 | 0.87 | 0.89 | 0.91 | 0.99 | 0.79 | 0.83 | 0.96 | 0.69 | 0.71 | 0.99 | 0.91 | 0.93 | 1 | 0.83 | 0.85 | 1.00 |
| | 1 | 0 | 0.05 | 0.05 | 0.05 | 0.04 | 0.06 | 0.04 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.06 | 0.05 | 0.06 | 0.06 |
| | | 7 | 0.24 | 0.25 | 0.46 | 0.29 | 0.33 | 0.74 | 0.25 | 0.28 | 0.61 | 0.23 | 0.24 | 0.75 | 0.30 | 0.33 | 0.94 | 0.27 | 0.30 | 0.87 |
| | | 13.5 | 0.41 | 0.43 | 0.86 | 0.56 | 0.60 | 0.99 | 0.47 | 0.51 | 0.96 | 0.44 | 0.46 | 0.99 | 0.60 | 0.64 | 1 | 0.52 | 0.56 | 1.00 |

Note: We drop the $(d_1)$ notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.

Table 6: Finite Sample Size and Size-adjusted Power for Joint Fractional Seasonal Variance Ratio Tests and HEGY Tests with Demeaning

| | | | $N = 100$ | | | | | | | | | $N = 400$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_1$ | $c$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.02 | 0.05 | 0.04 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.47 | 0.33 | 0.30 | 0.76 | 0.58 | 0.51 | 0.63 | 0.47 | 0.41 | 0.46 | 0.32 | 0.29 | 0.72 | 0.54 | 0.51 | 0.61 | 0.43 | 0.41 |
| | | 13.5 | 0.90 | 0.80 | 0.82 | 1 | 0.98 | 0.97 | 0.98 | 0.93 | 0.93 | 0.88 | 0.77 | 0.82 | 0.99 | 0.97 | 0.98 | 0.97 | 0.91 | 0.94 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.04 | 0.03 | 0.05 |
| i.i.d | 0.5 | 7 | 0.41 | 0.31 | 0.28 | 0.64 | 0.48 | 0.52 | 0.54 | 0.40 | 0.40 | 0.44 | 0.32 | 0.29 | 0.65 | 0.47 | 0.53 | 0.55 | 0.40 | 0.42 |
| | | 13.5 | 0.80 | 0.71 | 0.80 | 0.97 | 0.92 | 0.97 | 0.92 | 0.84 | 0.93 | 0.80 | 0.69 | 0.81 | 0.96 | 0.90 | 0.98 | 0.91 | 0.82 | 0.94 |
| | | 0 | 0.05 | 0.04 | 0.04 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.04 | 0.03 | 0.04 | 0.05 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 1 | 7 | 0.36 | 0.30 | 0.31 | 0.48 | 0.37 | 0.51 | 0.43 | 0.34 | 0.42 | 0.37 | 0.29 | 0.30 | 0.49 | 0.34 | 0.52 | 0.43 | 0.31 | 0.41 |
| | | 13.5 | 0.65 | 0.59 | 0.82 | 0.83 | 0.75 | 0.97 | 0.76 | 0.68 | 0.93 | 0.64 | 0.54 | 0.81 | 0.82 | 0.69 | 0.98 | 0.75 | 0.63 | 0.94 |
| | | 0 | 0.04 | 0.03 | 0.04 | 0.06 | 0.02 | 0.05 | 0.07 | 0.02 | 0.05 | 0.04 | 0.02 | 0.05 | 0.05 | 0.01 | 0.06 | 0.06 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.44 | 0.32 | 0.25 | 0.74 | 0.53 | 0.47 | 0.61 | 0.41 | 0.41 | 0.47 | 0.32 | 0.29 | 0.74 | 0.53 | 0.51 | 0.63 | 0.40 | 0.41 |
| | | 13.5 | 0.87 | 0.78 | 0.72 | 1 | 0.97 | 0.95 | 0.98 | 0.92 | 0.91 | 0.88 | 0.76 | 0.79 | 0.99 | 0.96 | 0.98 | 0.97 | 0.89 | 0.93 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 | 0.07 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 |
| MA(1): $\theta_1 = 0.5$ | 0.5 | 7 | 0.44 | 0.32 | 0.26 | 0.66 | 0.45 | 0.49 | 0.54 | 0.34 | 0.42 | 0.43 | 0.31 | 0.28 | 0.64 | 0.43 | 0.5 | 0.53 | 0.35 | 0.41 |
| | | 13.5 | 0.82 | 0.72 | 0.73 | 0.97 | 0.91 | 0.96 | 0.93 | 0.81 | 0.91 | 0.78 | 0.67 | 0.77 | 0.96 | 0.88 | 0.98 | 0.90 | 0.79 | 0.93 |
| | | 0 | 0.04 | 0.03 | 0.04 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 |
| | 1 | 7 | 0.39 | 0.30 | 0.26 | 0.52 | 0.33 | 0.48 | 0.45 | 0.28 | 0.41 | 0.36 | 0.28 | 0.28 | 0.49 | 0.31 | 0.51 | 0.41 | 0.27 | 0.43 |
| | | 13.5 | 0.68 | 0.58 | 0.73 | 0.86 | 0.72 | 0.95 | 0.78 | 0.62 | 0.90 | 0.63 | 0.54 | 0.78 | 0.83 | 0.68 | 0.98 | 0.74 | 0.58 | 0.94 |
| | | 0 | 0.07 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 | 0.07 | 0.02 | 0.05 | 0.06 | 0.02 | 0.04 | 0.05 | 0.01 | 0.04 | 0.06 | 0.01 | 0.04 |
| | 0.1 | 7 | 0.41 | 0.25 | 0.29 | 0.73 | 0.52 | 0.51 | 0.59 | 0.37 | 0.42 | 0.46 | 0.27 | 0.29 | 0.74 | 0.49 | 0.51 | 0.61 | 0.37 | 0.42 |
| | | 13.5 | 0.87 | 0.74 | 0.80 | 0.99 | 0.97 | 0.97 | 0.97 | 0.91 | 0.92 | 0.87 | 0.72 | 0.81 | 0.99 | 0.95 | 0.98 | 0.96 | 0.87 | 0.94 |
| | | 0 | 0.05 | 0.03 | 0.04 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.02 | 0.04 | 0.05 | 0.02 | 0.04 |
| AR(1): $\psi_1 = 0.5$ | 0.5 | 7 | 0.43 | 0.29 | 0.29 | 0.63 | 0.45 | 0.5 | 0.54 | 0.34 | 0.42 | 0.42 | 0.28 | 0.31 | 0.64 | 0.44 | 0.53 | 0.54 | 0.34 | 0.43 |
| | | 13.5 | 0.82 | 0.69 | 0.80 | 0.96 | 0.91 | 0.97 | 0.92 | 0.81 | 0.93 | 0.80 | 0.66 | 0.83 | 0.96 | 0.89 | 0.98 | 0.91 | 0.78 | 0.95 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.04 |
| | 1 | 7 | 0.39 | 0.28 | 0.30 | 0.5 | 0.34 | 0.5 | 0.42 | 0.28 | 0.43 | 0.38 | 0.28 | 0.31 | 0.47 | 0.34 | 0.52 | 0.42 | 0.29 | 0.43 |
| | | 13.5 | 0.69 | 0.57 | 0.81 | 0.84 | 0.73 | 0.97 | 0.77 | 0.63 | 0.94 | 0.66 | 0.55 | 0.82 | 0.82 | 0.71 | 0.98 | 0.75 | 0.61 | 0.94 |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.02 | 0.06 | 0.04 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 | 0.04 | 0.02 | 0.04 | 0.04 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.47 | 0.34 | 0.30 | 0.75 | 0.57 | 0.51 | 0.62 | 0.46 | 0.41 | 0.46 | 0.31 | 0.29 | 0.76 | 0.54 | 0.53 | 0.64 | 0.44 | 0.41 |
| | | 13.5 | 0.90 | 0.81 | 0.82 | 0.99 | 0.98 | 0.97 | 0.98 | 0.94 | 0.93 | 0.87 | 0.76 | 0.82 | 0.99 | 0.97 | 0.99 | 0.97 | 0.91 | 0.95 |
| AR(1): $\psi_1 = 0.5$ | | 0 | 0.05 | 0.03 | 0.05 | 0.04 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 0.5 | 7 | 0.43 | 0.33 | 0.29 | 0.66 | 0.47 | 0.5 | 0.55 | 0.40 | 0.41 | 0.42 | 0.31 | 0.31 | 0.64 | 0.45 | 0.5 | 0.54 | 0.39 | 0.40 |
| MA(1): $\theta_1 = 0.5$ | | 13.5 | 0.81 | 0.73 | 0.81 | 0.97 | 0.92 | 0.96 | 0.92 | 0.85 | 0.92 | 0.79 | 0.68 | 0.82 | 0.96 | 0.89 | 0.98 | 0.91 | 0.82 | 0.93 |
| | | 0 | 0.05 | 0.04 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 1 | 7 | 0.37 | 0.30 | 0.28 | 0.5 | 0.37 | 0.5 | 0.44 | 0.32 | 0.40 | 0.36 | 0.26 | 0.29 | 0.47 | 0.34 | 0.51 | 0.41 | 0.31 | 0.41 |
| | | 13.5 | 0.66 | 0.58 | 0.81 | 0.84 | 0.74 | 0.97 | 0.77 | 0.66 | 0.92 | 0.63 | 0.52 | 0.82 | 0.81 | 0.70 | 0.98 | 0.72 | 0.62 | 0.94 |
| | | 0 | 0.02 | 0.03 | 0.05 | 0.05 | 0.01 | 0.04 | 0.03 | 0.02 | 0.05 | 0.04 | 0.03 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.44 | 0.34 | 0.22 | 0.69 | 0.45 | 0.48 | 0.58 | 0.40 | 0.35 | 0.45 | 0.36 | 0.28 | 0.71 | 0.48 | 0.5 | 0.59 | 0.40 | 0.38 |
| | | 13.5 | 0.84 | 0.77 | 0.65 | 0.99 | 0.95 | 0.96 | 0.96 | 0.89 | 0.87 | 0.85 | 0.79 | 0.77 | 0.99 | 0.95 | 0.98 | 0.96 | 0.89 | 0.93 |
| | | 0 | 0.04 | 0.04 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 |
| AR(2): $\psi_2 = -0.5$ | 0.5 | 7 | 0.42 | 0.37 | 0.23 | 0.59 | 0.38 | 0.46 | 0.51 | 0.36 | 0.35 | 0.42 | 0.34 | 0.26 | 0.64 | 0.42 | 0.51 | 0.54 | 0.38 | 0.39 |
| | | 13.5 | 0.78 | 0.73 | 0.66 | 0.95 | 0.86 | 0.96 | 0.89 | 0.80 | 0.89 | 0.79 | 0.71 | 0.76 | 0.96 | 0.89 | 0.98 | 0.90 | 0.81 | 0.93 |
| | | 0 | 0.04 | 0.04 | 0.05 | 0.07 | 0.03 | 0.05 | 0.06 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.04 |
| | 1 | 7 | 0.36 | 0.30 | 0.25 | 0.45 | 0.29 | 0.49 | 0.41 | 0.27 | 0.38 | 0.36 | 0.29 | 0.27 | 0.47 | 0.32 | 0.51 | 0.42 | 0.29 | 0.40 |
| | | 13.5 | 0.63 | 0.56 | 0.69 | 0.81 | 0.68 | 0.96 | 0.74 | 0.61 | 0.89 | 0.63 | 0.55 | 0.77 | 0.81 | 0.69 | 0.98 | 0.75 | 0.62 | 0.93 |
| | | 0 | 0.01 | 0.03 | 0.06 | 0.01 | 0.02 | 0.06 | 0.01 | 0.03 | 0.05 | 0.03 | 0.04 | 0.06 | 0.03 | 0.03 | 0.07 | 0.03 | 0.03 | 0.07 |
| | 0.1 | 7 | 0.40 | 0.31 | 0.12 | 0.66 | 0.51 | 0.20 | 0.55 | 0.43 | 0.16 | 0.42 | 0.36 | 0.31 | 0.70 | 0.58 | 0.52 | 0.58 | 0.47 | 0.41 |
| | | 13.5 | 0.79 | 0.70 | 0.35 | 0.97 | 0.93 | 0.61 | 0.92 | 0.86 | 0.48 | 0.82 | 0.78 | 0.77 | 0.98 | 0.97 | 0.96 | 0.95 | 0.91 | 0.90 |
| MA(4): $\psi_4 = 0.5$ | | 0 | 0.03 | 0.04 | 0.06 | 0.02 | 0.03 | 0.05 | 0.02 | 0.03 | 0.05 | 0.04 | 0.03 | 0.06 | 0.04 | 0.03 | 0.07 | 0.04 | 0.03 | 0.06 |
| | 0.5 | 7 | 0.37 | 0.31 | 0.12 | 0.58 | 0.48 | 0.19 | 0.49 | 0.39 | 0.16 | 0.42 | 0.33 | 0.29 | 0.62 | 0.51 | 0.52 | 0.54 | 0.41 | 0.42 |
| MA(4): $\theta_4 = 0.5$ | | 13.5 | 0.71 | 0.65 | 0.36 | 0.93 | 0.88 | 0.62 | 0.85 | 0.79 | 0.51 | 0.78 | 0.69 | 0.75 | 0.95 | 0.91 | 0.97 | 0.90 | 0.82 | 0.91 |
| | | 0 | 0.03 | 0.04 | 0.05 | 0.03 | 0.03 | 0.05 | 0.03 | 0.04 | 0.05 | 0.05 | 0.04 | 0.06 | 0.05 | 0.03 | 0.08 | 0.04 | 0.03 | 0.07 |
| | 1 | 7 | 0.36 | 0.32 | 0.13 | 0.48 | 0.38 | 0.21 | 0.41 | 0.35 | 0.16 | 0.35 | 0.28 | 0.30 | 0.46 | 0.37 | 0.52 | 0.42 | 0.31 | 0.40 |
| | | 13.5 | 0.60 | 0.56 | 0.36 | 0.79 | 0.71 | 0.64 | 0.70 | 0.64 | 0.51 | 0.62 | 0.54 | 0.75 | 0.79 | 0.72 | 0.97 | 0.73 | 0.63 | 0.90 |

Note: We drop the ($d_1$) notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.

Table 7: Finite Sample Size and Size-adjusted Power for Joint Fractional Seasonal Variance Ratio Tests and HEGY Tests with Detrending

| | | | $N = 100$ | | | | | | | | | $N = 400$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d_1$ | $c$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ | $\tilde{\tau}_1$ | $\tilde{\tau}_1^{**}$ | $t_1$ | $\tau_{0S/2}$ | $\tau_{0S/2}^{**}$ | $F_{0S/2}$ | $\tau_{1S/2}$ | $\tau_{1S/2}^{**}$ | $F_{1S/2}$ |
| | | 0 | 0.04 | 0.03 | 0.04 | 0.03 | 0.02 | 0.06 | 0.03 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.06 | 0.05 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.20 | 0.12 | 0.18 | 0.35 | 0.17 | 0.27 | 0.28 | 0.14 | 0.22 | 0.19 | 0.09 | 0.16 | 0.33 | 0.14 | 0.26 | 0.27 | 0.12 | 0.21 |
| | | 13.5 | 0.58 | 0.44 | 0.57 | 0.88 | 0.72 | 0.82 | 0.77 | 0.59 | 0.72 | 0.56 | 0.37 | 0.53 | 0.85 | 0.65 | 0.82 | 0.75 | 0.54 | 0.70 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.04 | 0.02 | 0.06 | 0.04 | 0.02 | 0.06 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.06 | 0.05 | 0.02 | 0.05 |
| i.i.d | 0.5 | 7 | 0.21 | 0.12 | 0.17 | 0.31 | 0.19 | 0.27 | 0.26 | 0.16 | 0.22 | 0.19 | 0.13 | 0.16 | 0.29 | 0.16 | 0.25 | 0.24 | 0.14 | 0.21 |
| | | 13.5 | 0.54 | 0.40 | 0.57 | 0.80 | 0.66 | 0.82 | 0.70 | 0.55 | 0.72 | 0.5 | 0.38 | 0.53 | 0.77 | 0.59 | 0.82 | 0.65 | 0.49 | 0.70 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.06 | 0.05 | 0.02 | 0.06 | 0.05 | 0.03 | 0.06 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.06 |
| | 1 | 7 | 0.19 | 0.14 | 0.18 | 0.26 | 0.15 | 0.27 | 0.22 | 0.13 | 0.23 | 0.19 | 0.12 | 0.17 | 0.25 | 0.14 | 0.27 | 0.22 | 0.12 | 0.22 |
| | | 13.5 | 0.47 | 0.38 | 0.57 | 0.67 | 0.51 | 0.81 | 0.57 | 0.42 | 0.72 | 0.45 | 0.34 | 0.53 | 0.64 | 0.47 | 0.83 | 0.55 | 0.39 | 0.70 |
| | | 0 | 0.04 | 0.02 | 0.04 | 0.04 | 0.01 | 0.06 | 0.06 | 0.01 | 0.06 | 0.05 | 0.02 | 0.05 | 0.06 | 0.01 | 0.05 | 0.06 | 0.01 | 0.05 |
| | 0.1 | 7 | 0.21 | 0.11 | 0.15 | 0.36 | 0.15 | 0.25 | 0.28 | 0.11 | 0.22 | 0.21 | 0.1 | 0.16 | 0.32 | 0.13 | 0.26 | 0.27 | 0.1 | 0.22 |
| | | 13.5 | 0.61 | 0.43 | 0.47 | 0.89 | 0.70 | 0.76 | 0.79 | 0.54 | 0.67 | 0.58 | 0.36 | 0.5 | 0.85 | 0.62 | 0.81 | 0.76 | 0.48 | 0.69 |
| | | 0 | 0.04 | 0.03 | 0.05 | 0.06 | 0.01 | 0.06 | 0.07 | 0.02 | 0.06 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.06 | 0.01 | 0.05 |
| MA(1): $\theta_1 = 0.5$ | 0.5 | 7 | 0.21 | 0.13 | 0.15 | 0.30 | 0.14 | 0.25 | 0.25 | 0.12 | 0.21 | 0.19 | 0.12 | 0.16 | 0.31 | 0.14 | 0.26 | 0.25 | 0.11 | 0.22 |
| | | 13.5 | 0.55 | 0.40 | 0.46 | 0.80 | 0.59 | 0.76 | 0.69 | 0.47 | 0.66 | 0.49 | 0.37 | 0.49 | 0.76 | 0.55 | 0.80 | 0.65 | 0.43 | 0.69 |
| | | 0 | 0.05 | 0.03 | 0.04 | 0.06 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 |
| | 1 | 7 | 0.18 | 0.12 | 0.16 | 0.25 | 0.14 | 0.27 | 0.22 | 0.11 | 0.24 | 0.18 | 0.12 | 0.15 | 0.24 | 0.13 | 0.26 | 0.21 | 0.11 | 0.21 |
| | | 13.5 | 0.47 | 0.36 | 0.49 | 0.67 | 0.48 | 0.78 | 0.57 | 0.38 | 0.69 | 0.44 | 0.33 | 0.49 | 0.63 | 0.44 | 0.80 | 0.53 | 0.35 | 0.67 |
| | | 0 | 0.05 | 0.02 | 0.05 | 0.04 | 0.01 | 0.06 | 0.05 | 0.01 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.01 | 0.05 | 0.06 | 0.01 | 0.05 |
| | 0.1 | 7 | 0.21 | 0.1 | 0.18 | 0.35 | 0.15 | 0.27 | 0.29 | 0.11 | 0.23 | 0.22 | 0.09 | 0.17 | 0.33 | 0.14 | 0.25 | 0.27 | 0.11 | 0.21 |
| | | 13.5 | 0.61 | 0.40 | 0.55 | 0.89 | 0.68 | 0.80 | 0.79 | 0.53 | 0.72 | 0.60 | 0.36 | 0.53 | 0.85 | 0.63 | 0.81 | 0.74 | 0.48 | 0.68 |
| | | 0 | 0.06 | 0.02 | 0.05 | 0.05 | 0.02 | 0.06 | 0.06 | 0.02 | 0.05 | 0.05 | 0.02 | 0.05 | 0.05 | 0.01 | 0.05 | 0.05 | 0.02 | 0.05 |
| AR(1): $\psi_1 = 0.5$ | 0.5 | 7 | 0.19 | 0.11 | 0.17 | 0.29 | 0.15 | 0.24 | 0.24 | 0.11 | 0.23 | 0.19 | 0.1 | 0.16 | 0.30 | 0.13 | 0.25 | 0.24 | 0.11 | 0.21 |
| | | 13.5 | 0.55 | 0.39 | 0.56 | 0.80 | 0.62 | 0.79 | 0.70 | 0.49 | 0.73 | 0.51 | 0.33 | 0.52 | 0.78 | 0.55 | 0.81 | 0.67 | 0.43 | 0.70 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.03 | 0.05 | 0.05 | 0.03 | 0.05 |
| | 1 | 7 | 0.19 | 0.11 | 0.17 | 0.25 | 0.13 | 0.26 | 0.22 | 0.11 | 0.23 | 0.17 | 0.11 | 0.17 | 0.24 | 0.14 | 0.25 | 0.20 | 0.12 | 0.22 |
| | | 13.5 | 0.48 | 0.33 | 0.56 | 0.67 | 0.48 | 0.80 | 0.58 | 0.39 | 0.72 | 0.44 | 0.33 | 0.55 | 0.63 | 0.48 | 0.81 | 0.53 | 0.39 | 0.71 |
| | | 0 | 0.03 | 0.02 | 0.05 | 0.03 | 0.02 | 0.06 | 0.03 | 0.02 | 0.06 | 0.04 | 0.02 | 0.05 | 0.04 | 0.01 | 0.05 | 0.04 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.21 | 0.12 | 0.17 | 0.35 | 0.18 | 0.26 | 0.28 | 0.15 | 0.22 | 0.21 | 0.11 | 0.16 | 0.35 | 0.16 | 0.27 | 0.28 | 0.14 | 0.21 |
| | | 13.5 | 0.62 | 0.44 | 0.57 | 0.89 | 0.74 | 0.81 | 0.80 | 0.61 | 0.71 | 0.58 | 0.41 | 0.54 | 0.86 | 0.67 | 0.83 | 0.76 | 0.56 | 0.71 |
| AR(1): $\psi_1 = 0.5$ | | 0 | 0.04 | 0.03 | 0.05 | 0.04 | 0.02 | 0.06 | 0.04 | 0.02 | 0.06 | 0.05 | 0.03 | 0.05 | 0.05 | 0.02 | 0.05 | 0.06 | 0.02 | 0.05 |
| | 0.5 | 7 | 0.20 | 0.12 | 0.16 | 0.31 | 0.16 | 0.27 | 0.26 | 0.14 | 0.22 | 0.18 | 0.12 | 0.16 | 0.29 | 0.17 | 0.26 | 0.23 | 0.14 | 0.21 |
| MA(1): $\theta_1 = 0.5$ | | 13.5 | 0.56 | 0.41 | 0.57 | 0.82 | 0.65 | 0.82 | 0.71 | 0.53 | 0.73 | 0.5 | 0.38 | 0.53 | 0.76 | 0.60 | 0.82 | 0.64 | 0.48 | 0.70 |
| | | 0 | 0.05 | 0.03 | 0.05 | 0.04 | 0.02 | 0.06 | 0.04 | 0.02 | 0.05 | 0.05 | 0.03 | 0.06 | 0.05 | 0.02 | 0.05 | 0.05 | 0.02 | 0.06 |
| | 1 | 7 | 0.18 | 0.12 | 0.16 | 0.27 | 0.15 | 0.26 | 0.22 | 0.13 | 0.22 | 0.19 | 0.12 | 0.16 | 0.26 | 0.14 | 0.26 | 0.22 | 0.13 | 0.21 |
| | | 13.5 | 0.48 | 0.38 | 0.58 | 0.69 | 0.53 | 0.83 | 0.59 | 0.44 | 0.73 | 0.45 | 0.33 | 0.52 | 0.63 | 0.45 | 0.82 | 0.54 | 0.39 | 0.71 |
| | | 0 | 0.01 | 0.03 | 0.06 | 0.03 | 0.01 | 0.05 | 0.02 | 0.02 | 0.05 | 0.03 | 0.03 | 0.05 | 0.05 | 0.01 | 0.05 | 0.05 | 0.02 | 0.05 |
| | 0.1 | 7 | 0.19 | 0.14 | 0.14 | 0.34 | 0.13 | 0.25 | 0.27 | 0.13 | 0.20 | 0.20 | 0.13 | 0.16 | 0.33 | 0.12 | 0.25 | 0.25 | 0.12 | 0.20 |
| | | 13.5 | 0.54 | 0.45 | 0.41 | 0.86 | 0.63 | 0.77 | 0.74 | 0.54 | 0.64 | 0.56 | 0.45 | 0.5 | 0.86 | 0.59 | 0.81 | 0.73 | 0.51 | 0.68 |
| | | 0 | 0.02 | 0.04 | 0.06 | 0.05 | 0.01 | 0.05 | 0.04 | 0.02 | 0.05 | 0.04 | 0.03 | 0.05 | 0.05 | 0.01 | 0.05 | 0.05 | 0.02 | 0.05 |
| AR(2): $\psi_2 = -0.5$ | 0.5 | 7 | 0.19 | 0.16 | 0.13 | 0.30 | 0.12 | 0.25 | 0.24 | 0.12 | 0.19 | 0.20 | 0.14 | 0.15 | 0.28 | 0.13 | 0.26 | 0.24 | 0.13 | 0.20 |
| | | 13.5 | 0.5 | 0.45 | 0.39 | 0.79 | 0.54 | 0.76 | 0.65 | 0.46 | 0.61 | 0.5 | 0.40 | 0.48 | 0.75 | 0.52 | 0.81 | 0.64 | 0.45 | 0.67 |
| | | 0 | 0.03 | 0.04 | 0.05 | 0.05 | 0.01 | 0.05 | 0.04 | 0.02 | 0.05 | 0.04 | 0.03 | 0.05 | 0.06 | 0.02 | 0.04 | 0.05 | 0.02 | 0.04 |
| | 1 | 7 | 0.20 | 0.17 | 0.14 | 0.26 | 0.11 | 0.25 | 0.22 | 0.12 | 0.20 | 0.19 | 0.13 | 0.16 | 0.24 | 0.12 | 0.27 | 0.21 | 0.12 | 0.21 |
| | | 13.5 | 0.46 | 0.41 | 0.42 | 0.66 | 0.41 | 0.78 | 0.56 | 0.38 | 0.64 | 0.44 | 0.35 | 0.49 | 0.61 | 0.42 | 0.82 | 0.53 | 0.36 | 0.69 |
| | | 0 | 0.00 | 0.02 | 0.04 | 0 | 0.02 | 0.04 | 0 | 0.02 | 0.04 | 0.02 | 0.03 | 0.08 | 0.01 | 0.03 | 0.11 | 0.02 | 0.03 | 0.10 |
| | 0.1 | 7 | 0.18 | 0.1 | 0.09 | 0.30 | 0.14 | 0.12 | 0.24 | 0.12 | 0.11 | 0.19 | 0.13 | 0.15 | 0.30 | 0.21 | 0.23 | 0.23 | 0.16 | 0.20 |
| | | 13.5 | 0.47 | 0.32 | 0.20 | 0.77 | 0.57 | 0.36 | 0.66 | 0.46 | 0.29 | 0.53 | 0.43 | 0.43 | 0.81 | 0.72 | 0.71 | 0.69 | 0.58 | 0.59 |
| AR(4): $\psi_4 = 0.5$ | | 0 | 0.01 | 0.04 | 0.04 | 0 | 0.03 | 0.04 | 0.01 | 0.03 | 0.04 | 0.03 | 0.04 | 0.07 | 0.02 | 0.03 | 0.1 | 0.03 | 0.03 | 0.09 |
| | 0.5 | 7 | 0.18 | 0.15 | 0.09 | 0.28 | 0.20 | 0.13 | 0.23 | 0.18 | 0.12 | 0.19 | 0.14 | 0.17 | 0.29 | 0.20 | 0.24 | 0.23 | 0.17 | 0.20 |
| MA(4): $\theta_4 = 0.5$ | | 13.5 | 0.44 | 0.39 | 0.21 | 0.71 | 0.61 | 0.37 | 0.59 | 0.52 | 0.30 | 0.5 | 0.41 | 0.45 | 0.74 | 0.64 | 0.71 | 0.63 | 0.54 | 0.59 |
| | | 0 | 0.02 | 0.04 | 0.03 | 0.01 | 0.03 | 0.04 | 0.02 | 0.04 | 0.04 | 0.04 | 0.04 | 0.08 | 0.03 | 0.03 | 0.1 | 0.04 | 0.04 | 0.09 |
| | 1 | 7 | 0.18 | 0.16 | 0.09 | 0.24 | 0.18 | 0.13 | 0.20 | 0.17 | 0.11 | 0.19 | 0.15 | 0.15 | 0.25 | 0.18 | 0.25 | 0.22 | 0.16 | 0.20 |
| | | 13.5 | 0.42 | 0.38 | 0.23 | 0.60 | 0.52 | 0.37 | 0.5 | 0.45 | 0.31 | 0.43 | 0.37 | 0.41 | 0.62 | 0.53 | 0.70 | 0.52 | 0.45 | 0.57 |

Note: We drop the $(d_1)$ notation. The bootstrap results are obtained with Fast double bootstrap algorithm. For HEGY we apply optimal lag lenght selection routine of del Barrio Castro et al. (2016). For the HEGY type tests, we also rerun the simulations for different $d_1$s. Since this parameter does not appear in HEGY testing, results are similar. The small differences may exist because of simulation uncertainty and small sample.